$$f(\nu_m) = \mathrm{Const}\ K\ \theta_s^3\ B^{\frac{\gamma+1}{2}}\ \nu_m^{-\left(\frac{\gamma-1}{2}\right)}$$

$$= \mathrm{Const}\left[K_0\left(\frac{r}{r_0}\right)^{-(\gamma+2)}\right]\left[\theta_0\left(\frac{r}{r_0}\right)\right]^3\left[B_0\left(\frac{r}{r_0}\right)^{-1}\right]^{\frac{\gamma+1}{2}} \times$$

$$\left[\nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)}\right]^{-\left(\frac{\gamma-1}{2}\right)}$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\left(\frac{\gamma+1}{2}\right)}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)} \times \left[\left(\frac{r}{r_0}\right)^{-(\gamma+2)}\left(\frac{r}{r_0}\right)^3\left(\frac{r}{r_0}\right)^{-(\gamma+1)} \times \left(\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)}\right)^{-\left(\frac{\gamma-1}{2}\right)}\right]$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\frac{\gamma+1}{2}}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)} \times \left[\left(\frac{r}{r_0}\right)^{-\gamma-2+3-\gamma-1+\frac{(4\gamma+6)(\gamma-1)}{2(\gamma+4)}}\right]$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\frac{\gamma+1}{2}}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)}\left[\left(\frac{r}{r_0}\right)^{-2\gamma+\frac{(4\gamma+6)(\gamma-1)}{2(\gamma+4)}}\right]$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\frac{\gamma+1}{2}}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)}\left[\left(\frac{r}{r_0}\right)^{\frac{-2\gamma\times 2(\gamma+4)+(4\gamma+6)(\gamma-1)}{2(\gamma+4)}}\right]$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\frac{\gamma+1}{2}}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)}\left[\left(\frac{r}{r_0}\right)^{\frac{-4\gamma^2-16\gamma+4\gamma^2+2\gamma-6}{2(\gamma+4)}}\right]$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\frac{\gamma+1}{2}}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)}\left(\frac{r}{r_0}\right)^{\frac{-14\gamma-6}{2(\gamma+4)}}$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\frac{\gamma+1}{2}}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)}\left(\frac{r}{r_0}\right)^{\frac{-2(7\gamma+3)}{2(\gamma+4)}}$$

$$= \mathrm{Const}\ K_0\ \theta_0^3\ B_0^{\frac{\gamma+1}{2}}\ \nu_{m0}^{-\left(\frac{\gamma-1}{2}\right)}\left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)}$$

Therefore

$$F(\nu_m) = F(\nu_{mo}) \left(\frac{r}{r_o}\right)^{-\left(\frac{7\delta+3}{\delta+4}\right)}$$

with

$$F(\nu_{mo}) = \text{Const } K_o \, \theta_o^3 \, B_o^{\frac{\delta+1}{2}} \, \nu_{mo}^{-\left(\frac{\delta-1}{2}\right)}$$

$$\Rightarrow \quad F(\nu_m) = F(\nu_{mo}) \left(\frac{r}{r_o}\right)^{-\left(\frac{7\delta+3}{\delta+4}\right)}$$

$$= F_{mo} \left(\frac{r}{r_o}\right)^{-\left(\frac{7\delta+3}{\delta+4}\right)}$$

But this is not the end result. We have to keep in mind that this flux is emitted in a cloud that contains a thermal plasma. If the cloud is optically thick the flux density scale with

$$F(\nu_s) = 2\,\theta_s^2 \, \nu_L^{-1/2} \, \nu^{5/2}$$

This result was obtained earlier when the kinetic temperature of the relativistic electrons was put equal to kinetic temperature of the thermal plasma.

We know that the brightness (intensity) of a emitting region scale as

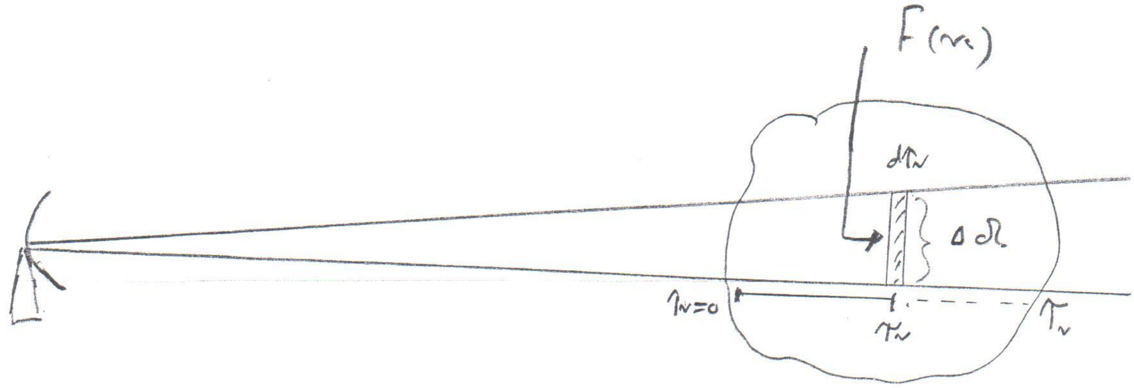
$$I_\nu(\tau_\nu) = S_\nu (1 - e^{-\tau_\nu})$$

$$F_\nu(\tau_\nu) = I_\nu(\tau_\nu)\,\Delta\Omega = \underbrace{S_\nu \Delta\Omega}\,(1 - e^{-\tau_\nu})$$

$$F_\nu(\tau_\nu) = F(\nu_s)\,(1 - e^{-\tau_\nu})$$

$$F_\nu(\tau_\nu) \propto \nu^{5/2}(1 - e^{-\tau_\nu})$$

Let's explain this



The observed flux for an optically thick source is the difference between the flux produced inside ($F(\nu_s)$) and the flux that is absorbed along the way $F(\nu_s)e^{-\tau_\nu}$.

With this one can write

$$\frac{F(\nu, r)}{F(\nu_m, r)} = \left(\frac{\nu}{\nu_m}\right)^{5/2} \left[\frac{1 - e^{-\tau(\nu)}}{1 - e^{-\tau(\nu_m)}}\right]$$

Then

$$F(\nu, r) = F(\nu_m, r) \left(\frac{\nu}{\nu_m}\right)^{5/2} \left[\frac{1 - e^{-\tau(\nu)}}{1 - e^{-\tau(\nu_m)}}\right]$$

$$= \left[F(\nu_{m0}) \left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)}\right] \left(\frac{\nu}{\nu_m}\right)^{5/2} \left[\frac{1 - e^{-\tau(\nu)}}{1 - e^{-\tau(\nu_m)}}\right]$$

$$= F(\nu_{m0}) \left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)} \nu^{5/2} \times \left[\nu_{m0} \left(\frac{r}{r_0}\right)^{-\left(\frac{6\gamma+6}{\gamma+4}\right)}\right]^{-5/2}$$

$$\times \left[\frac{1 - e^{-\tau(\nu)}}{1 - e^{-\tau(\nu_m)}}\right]$$

$$= F(\nu_{m0}) \left(\frac{\nu}{\nu_{m0}}\right)^{5/2} \left[\left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)} \left(\frac{r}{r_0}\right)^{\frac{5}{2}\left(\frac{6\gamma+6}{\gamma+4}\right)}\right]$$

$$\times \left[\frac{1 - e^{-\tau(\nu)}}{1 - e^{-\tau(\nu_m)}}\right]$$

We need to find a scaling for this i.t.o frequency.

Before we solve this equation we need to get an expression for the optical depth $\tau(\nu)$. The optical depth

$$\tau(\nu) = \int \alpha(\nu)\, dl$$

But the absorption coefficient can also be expressed in terms of the ratio between the volume emissivity and the brightness

$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

$$\Rightarrow \quad \alpha_\nu = \frac{j_\nu}{S_\nu}$$

$$\propto \frac{K\, B^{\left(\frac{\gamma+1}{2}\right)}\, \nu^{-\left(\frac{\gamma-1}{2}\right)}}{B^{-1/2}\, \nu^{5/2}}$$

$$\propto K \left(B^{\frac{\gamma+1}{2}+\frac{1}{2}}\right) \nu^{-\left(\frac{\gamma-1}{2}\right)-\frac{5}{2}}$$

$$\propto K\, B^{\frac{\gamma+2}{2}}\, \nu^{-\left(\frac{\gamma+4}{2}\right)}$$

$$\alpha_\nu = \text{Const}\; K\; B^{\left(\frac{\gamma+2}{2}\right)}\; \nu^{-\left(\frac{\gamma+4}{2}\right)}$$

$$\tau(\nu) = \alpha_\nu\, l = \text{Const}\; K\; B^{\left(\frac{\gamma+2}{2}\right)}\; \nu^{-\left(\frac{\gamma+4}{2}\right)}\, l$$

$$\tau(\nu) \propto \nu^{-\left(\frac{\gamma+4}{2}\right)}$$

$$\tau(\nu_m) \propto \nu_m^{-\left(\frac{\gamma+4}{2}\right)}$$

$$\Rightarrow \quad \frac{\tau(\nu)}{\tau(\nu_m)} = \left(\frac{\nu}{\nu_m}\right)^{-\left(\frac{\gamma+4}{2}\right)}$$

$$\tau(\nu) = \tau(\nu_m)\left(\frac{\nu}{\nu_m}\right)^{-\left(\frac{\gamma+4}{2}\right)}$$

This can now be inserted in the expression for the flux density

$$F(\nu, r) = F(\nu_{mo})\left(\frac{\nu}{\nu_{mo}}\right)^{5/2}\left[\left(\frac{r}{r_o}\right)^{-\frac{2(7\gamma+3)+5(4\gamma+6)}{2(\gamma+4)}}\right]$$

$$\times\left[\frac{1-e^{-\tau(\nu)}}{1-e^{-\tau(\nu_m)}}\right]$$

Using the notation:

$$F(\nu_{mo}) = F_{mo}$$

$$f(\nu,r) = f_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{r}{r_0}\right)^{\frac{-14\delta-6+20\delta+30}{2(\delta+4)}}$$

$$\times\ \frac{1-e^{-\left[\tau(\nu_m)\left(\frac{\nu}{\nu_m}\right)^{-\left(\frac{\delta+4}{2}\right)}\right]}}{1-e^{-\tau(\nu_m)}}$$

$$= f_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{r}{r_0}\right)^{\frac{6\delta+24}{2(\delta+4)}}\ \times$$

$$\frac{1-e^{-\tau(\nu_m)\left(\frac{\nu}{\nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\delta+6}{\delta+4}\right)}}\right)^{-\left(\frac{\delta+4}{2}\right)}}}{1-e^{-\tau(\nu_m)}}$$

$$= f_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{r}{r_0}\right)^{\frac{2(3\delta+12)}{2(\delta+4)}}\ \times$$

$$\frac{1-e^{-\left[\tau(\nu_m)\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\delta+6}{\delta+4}\right)\left(\frac{\delta+4}{2}\right)}\right]}}{1-e^{-\tau(\nu_m)}}$$

$$= f_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{r}{r_0}\right)^{\overbrace{\frac{3\delta+12}{\delta+4}}^{\frac{3(\delta+4)}{(\delta+4)}}}\ \times$$

$$\frac{1-e^{-\left[\tau(\nu_m)\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)}\left(\frac{r}{r_0}\right)^{\overbrace{-\left(\frac{4\delta+6}{2}\right)}^{-[2\delta+3]}}\right]}}{1-e^{-\tau(\nu_m)}}$$

This reduces to

$$F(\nu, r) = F_{m0} \left(\frac{\nu}{\nu_{m0}}\right)^{5/2} \left(\frac{r}{r_0}\right)^{3} \times \left[\frac{1 - e^{-\left[\tau(\nu_m)\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)}\left(\frac{r}{r_0}\right)^{-(2\delta+3)}\right]}}{1 - e^{-\tau(\nu_m)}}\right]$$

Here F_{m0} is the maximum flux density at a time t_0 with ν_{m0} the corresponding frequency at this maximum and where $\tau(\nu_m)$ represents the optical depth at this maximum.

The optical dept $\tau(\nu)$ is frequency dependent but most of the emission is emitted where

$$\tau(\nu) = \tau(\nu_m) = 1$$

This means that one can probe into the source as was indicated earlier. For optical depths $\tau(\nu) \gg 1$ one can envisage probing only a thin surface layer of the source with thickness

$$\Delta R = \frac{R}{\tau(\nu)} \qquad \tau(\nu) \gg 1$$

This can be explained as follows:

$$I(\nu) = S(\nu)\left(1 - e^{-\tau(\nu)}\right)$$

$$= S(\nu) \qquad \tau(\nu) \gg 1$$

$$F(\nu) = I(\nu)\,\Delta\Omega = S(\nu)\,\Delta\Omega$$

$$\Rightarrow F(\nu) = S(\nu)\,\Delta\Omega$$

$$= \frac{j(\nu)\,\Delta\Omega}{\alpha(\nu)}$$

$$= \frac{j(\nu)\,\Delta\Omega\, R}{\alpha(\nu)\, R}$$

$$= \frac{j(\nu)\,\Delta\Omega\, R}{\tau(\nu)}$$

$$= j(\nu)\,\frac{R}{\tau(\nu)}\,\Delta\Omega$$

$$= j(\nu)\,\Delta R\,\Delta\Omega$$

$$\Delta R = \frac{R}{\tau(\nu)}$$

Effectively one see only radiation from the thin shell on the edge of the source which is optically thin

We can now investigate the eqn for the observed flux density. Let's see the asymptotic limits for $\nu \ll \nu_m$ and $\nu \gg \nu_m$.

- $\nu \ll \nu_m$ or $\nu \ll \nu_{m0}$: $\left(\frac{\nu}{\nu_{m0}}\right) \to 0$

In this case the exponential term in the numerator vanishes

$$F(\nu, r) \approx F_{m0} \left(\frac{\nu}{\nu_{m0}}\right)^{5/2} \left(\frac{r}{r_0}\right)^3 \quad \text{since} \left\{ \frac{\left[1 - e^{-\left[\tau_m \left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)} \cdots\right]}\right] \to 1 \; (*)}{1 - e^{-\tau(\nu_m)} \to (1 - e^{-1})} \right\}$$

$$\approx F_0(\nu) \left(\frac{r}{r_0}\right)^3$$

with

$$\underbrace{F_0(\nu) = F_{m0} \left(\frac{\nu}{\nu_{m0}}\right)^{5/2}}_{\text{Source is optically thick}}$$

$*$ Notice that for $\left(\frac{\nu}{\nu_{m0}}\right) \to 0$ we have that $\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)} \to \infty$ so that the exponential vanishes in the numerator

For the other case

• $\nu \gg \nu_m$ or $\nu \gg \nu_{m0}$:

Here $\tau(\nu_m)\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)}\left(\frac{r}{r_0}\right)^{-(2\delta+3)} \rightarrow$ small

Then the numerator with x small is

$$1 - e^{-x} = 1 - [1 - x + HOT]$$

$$= x \quad [\because x \text{ is small}]$$

Therefore the flux can be written as follows:

$$F(\nu, r) = F_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{r}{r_0}\right)^{3} \frac{\tau(\nu_m)\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)}\left(\frac{r}{r_0}\right)^{-(2\delta+3)}}{1 - e^{-\tau(\nu_m)}}$$

$$\approx F_{m0}\left[\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta+4}{2}\right)}\right]\left[\left(\frac{r}{r_0}\right)^{3}\left(\frac{r}{r_0}\right)^{-(2\delta+3)}\right]$$

$$\approx F_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{\frac{5-\delta-4}{2}}\left(\frac{r}{r_0}\right)^{-2\delta}$$

$$\approx F_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta-1}{2}\right)}\left(\frac{r}{r_0}\right)^{-2\delta}$$

$$= F_0(\nu)\left(\frac{r}{r_0}\right)^{-2\delta} \quad \text{with} \quad F_0(\nu) = F_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\delta-1}{2}\right)}$$

The influence of expanding on absorption

For electromagnetic radiation propagating through a plasma with electron density n_e, all emission below a critical frequency will be surpressed. This frequency is the so-called Razin frequency

$$\nu_R \approx \frac{20 n_e}{B} \qquad \text{(Razin 1960)}$$

But we know that

$$n_e = \frac{N_e}{V}$$

$$\propto \frac{N_e}{r^3}$$

$$n_0 \propto \frac{N_e}{r_0^3}$$

$$\Rightarrow \quad \frac{n_e(r)}{n_e(r_0)} = \left(\frac{r}{r_0}\right)^{-3}$$

$$\Rightarrow \quad n_e(r) = n_e(r_0)\left(\frac{r}{r_0}\right)^{-3}$$

We also showed that the magnetic field in an expanding source evolves

$$B = B_0 \left(\frac{r}{r_0}\right)^{-2}$$

Therefore

$$\nu_R(r) = \frac{20\, n_e(r)}{B(r)}$$

$$= \frac{20\, n_e(r_0) \left(\frac{r}{r_0}\right)^{-3}}{B_0 \left(\frac{r}{r_0}\right)^{-2}}$$

$$= \frac{20\, n_e(r_0)}{B(r_0)} \left(\frac{r}{r_0}\right)^{-1}$$

$$= \nu_R(r_0) \left(\frac{r}{r_0}\right)^{-1}$$

So for an expanding source the frequency below which the emission will be surpressed shifts towards lower values resulting in the source becoming increasingly optically thin, i.e transparent to its own radiation

Dynamic evolution of a blob

The relation that describes the radius of a blob vs expansion time is

$$\left(\frac{r}{r_0}\right) = \left(1 + \frac{t}{\beta t_0}\right)^{\beta}$$

where t_0 is determined from

$$\frac{\dot{r}_0}{r_0} = \frac{1}{t_0} \quad \Rightarrow \quad t_0 = \frac{r_0}{\dot{r}_0}$$

$$\left(\frac{r}{r_0}\right) = \left(1 + \frac{t\,\dot{r}_0}{\beta r_0}\right)^{\beta}$$

Here β is a constant that depends on the property of the medium the blob is expanding into, i.e $\beta = 1$ (constant expansion velocity), $\beta = 2/5$ (expansion into a uniform medium) and $\beta = \frac{2}{3}$ (expansion into a stellar wind).

One can get asymptotic limits of the equation above for small and large t.

. t small: Expand into series $(a+x)^n = a^n + na^{n-1}x + HOT$

$$\left(\frac{r}{r_0}\right) = \left(1 + \frac{t\,V_0}{\beta r_0}\right)^{\beta}$$

$$\boxed{\begin{aligned} a &= 1 \\ x &= \frac{Vt_0}{\beta r_0} \\ \beta &= n \end{aligned}}$$

$$= \left(1 + \left(\frac{V_0 t}{\beta r_0}\right)\right)^{\beta}$$

$$\simeq 1 + \beta\left(\frac{V_0 t}{\beta r_0}\right) + HOT$$

$$\approx 1 + \frac{V_0 t}{r_0}$$

$$\left(\frac{r}{r}\right) \simeq 1 + \frac{t}{\left(\frac{r_0}{V_0}\right)} \approx 1 + \frac{t}{t_0} \approx \frac{t}{t_0}$$

. t large: Second term dominates.

$$\left(\frac{r}{r_0}\right) = \left(1 + \frac{V_0 t}{\beta r_0}\right)^{\beta}$$

$$\approx \left(\frac{V_0 t}{\beta r_0}\right)^{\beta}$$

$$\simeq \left(\frac{t}{\beta \frac{r_0}{V_0}}\right)^{\beta}$$

$$\approx \left(\frac{t}{\beta t_0}\right)^{\beta}$$

Expansion into a gaseous medium

For a spherical electron cloud which expands the internal pressure decreases very rapidly which will slow down the expansion velocity. For a thermal gas the thermal pressure is

$$P_{th} = \frac{2}{3} \frac{E_{th}}{V}$$

The pressure in relativistic electrons

$$P_{rel} = \frac{1}{3} \frac{E_{rel}}{V}$$

The total pressure at any given radial distance is then

Relativistic particle energy.

$$P(r) = P_{rel} + P_{th}$$

The termal energy is the converted relativistic particle energy as the cloud expands into the medium.

$$= \frac{1}{3} \frac{E_{rel}}{V} + \frac{2}{3} \frac{E_{th}}{V}$$

$$= \frac{1}{3} \frac{E_{rel}}{V} + \frac{2}{3} \left(\frac{E_0 - E_{rel}}{V} \right)$$

Relativistic energy at time t_0

$$= \frac{1}{3} \frac{E_{rel}}{V} + \frac{2}{3} \frac{E_0}{V} - \frac{2}{3} \frac{E_{rel}}{V}$$

Initial relativistic energy

Relativistic energy at time t.

$$P(r) = \frac{2}{3}\frac{E_0}{V} - \frac{1}{3}\frac{E_{rel}}{V}$$

We showed this earlier

But

$$E_{rel} \approx E_0 \left(\frac{r}{r_0}\right)^{-1}$$

$$P(r) \approx \frac{2}{3}\frac{E_0}{V} - \frac{1}{3}\frac{E_0\left(\frac{r}{r_0}\right)^{-1}}{V}$$

$$\approx \frac{1}{3}\frac{E_0}{V}\left[2 - \left(\frac{r}{r_0}\right)^{-1}\right]$$

For $r \gg r_0$

$$P(r) \approx \frac{2}{3}\frac{E_0}{V}$$

[The relativistic particles thermalized - cooled significantly]

$$\approx \frac{2}{3}\frac{E_0}{\frac{4\pi}{3}r^3}$$

$$\approx \frac{E_0}{2\pi r^3}$$

This expression shows the pressure the expanding bubble exerts on the medium into which it expands. The ram pressure of the expanding cloud

$$\rho_0\left(\frac{dr}{dt}\right)^2 \approx \frac{E_0}{2\pi r^3}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{E_0}{2\pi\rho_0 r^3}$$

$$\left(\frac{dr}{dt}\right) = \left(\frac{E_0}{2\pi\rho_0}\right)^{1/2} r^{-3/2}$$

$$r^{3/2}\,dr = \left(\frac{E_0}{2\pi\rho_0}\right)^{1/2} dt$$

$$\int_{r_0}^{r} r^{3/2}\,dr = \left(\frac{E_0}{2\pi\rho_0}\right)^{1/2} \int_0^t dt$$

$$\left[\frac{r^{5/2}}{5/2}\right]_{r_0}^{r} = \left(\frac{E_0}{2\pi\rho_0}\right)^{1/2} t$$

$$\frac{2}{5}\left[r^{5/2} - r_0^{5/2}\right] = \left(\frac{E_0}{2\pi\rho_0}\right)^{1/2} t$$

for $r \gg r_0$

$$\frac{2}{5} r^{5/2} = \left(\frac{E_0}{2\pi\rho_0}\right)^{1/2} t$$

$$t = \frac{2}{5}\left(\frac{2\pi\rho_0}{E_0}\right)^{1/2} r^{5/2}$$

$$= \frac{2(2\pi)^{1/2}}{5}\left(\frac{\rho_0}{E_0}\right)^{1/2} r^{5/2}$$

The expansion time is then

$$t_{exp} = \left(\frac{\rho_0}{E_0}\right)^{1/2} r^{5/2}$$

$$= \left(\frac{\rho_0 \, r_0^5}{E_0}\right)^{1/2} \left(\frac{r}{r_0}\right)^{5/2}$$

Notice that

$$\left[\begin{array}{l} (2\pi)^{1/2} \approx 5/2 \\ \Rightarrow \frac{2}{5} \times \frac{5}{2} \approx 1 \end{array}\right]$$

However, one can also write using the result on the previous page:

$$t_{exp} = \frac{(2\pi)^{1/2}}{5/2} \left(\frac{\rho_0}{E_0}\right)^{1/2} r^{5/2}$$

$$= \left(\frac{2\pi \rho_0}{E_0}\right)^{1/2} \frac{1}{5/2} r^{5/2}$$

$$= \left(\frac{2\pi \rho_0}{E_0}\right)^{1/2} \frac{1}{5/2} r_0^{5/2} \left(\frac{r}{r_0}\right)^{5/2}$$

$$= \overset{t_0 \,=}{\underbrace{\left(\frac{2\pi \rho_0 r_0^5}{E_0}\right)^{1/2}}} \frac{1}{5/2} \left(\frac{r}{r_0}\right)^{5/2}$$

$$\hookrightarrow \left[\frac{\rho_0 (g\,cm^{-3}) \; r_0^5 (cm^5)}{g\,cm^2\,s^{-2}}\right]^{1/2} = (sec)$$

$$\left(\frac{t_{exp}}{t_0}\right) = \frac{1}{(5/2)} \left(\frac{r}{r_0}\right)^{5/2}$$

$$\Rightarrow \left(\frac{r}{r_0}\right)^{5/2} = \frac{5}{2} \left(\frac{t_{exp}}{t_0}\right)$$

$$\left(\frac{r}{r_0}\right)^{5/2} = \left(\frac{t_{exp}}{\frac{2}{5}\, t_0}\right)$$

$$\left(\frac{r}{r_0}\right) = \left(\frac{t_{exp}}{\frac{2}{5}\, t_0}\right)^{2/5}$$

Here $\beta = \frac{2}{5}$ $\Rightarrow$ $\left(\frac{r}{r_0}\right) = \left(\frac{t_{exp}}{\beta\, t_0}\right)^{\beta}$

How will this relation influences the evolution of the flux density?

We show earlier that

$\nu \ll \nu_m$:
$$F(\nu, r) = F_0(\nu) \left(\frac{r}{r_0}\right)^3$$
$$= F_0(\nu) \left[\left(\frac{t_{exp}}{\beta t_0}\right)^{\beta}\right]^3$$
$$\propto F_0(\nu)\, t_{exp}^{3} \quad [\beta \to 1 \text{ initially}]$$

$\nu \gg \nu_m$:
$$F(\nu, r) = F_0(\nu) \left(\frac{r}{r_0}\right)^{-2\gamma}$$
$$= F_0(\nu) \left[\left(\frac{t_{exp}}{\beta t_0}\right)^{\beta}\right]^{-2\gamma}$$
$$\propto F_0(\nu)\, t_{exp}^{-2\beta\gamma} \quad [\beta < 1]$$

If the rise of the flare flux density is steeper than the decay time the slope on the rise must also be steeper. Then

$$\Rightarrow \quad 3 > 2\beta\gamma$$

- $\beta \to 1$: $\gamma < \frac{3}{2}$ (Hard spectrum)

- $\beta < 1$: $3 > 2\beta\gamma$

$$\gamma < \frac{3}{2\beta}$$

$$\gamma > \frac{3}{2} \quad \text{(Soft spectrum)}$$

This allows one to constrain the electron spectrum of an expanding synchrotron emitting cloud. For $\beta < 1$ the blob decelerate into the surrounding medium. The definition of β will be quantified in the following section.

Time Evolution of Flares

It was shown that for individual flares

$$\nu \ll \nu_m : \quad F(\nu) = F_0(\nu)\left(\frac{r}{r_0}\right)^{3}$$

$$\nu \gg \nu_m : \quad F(\nu) = F_0(\nu)\left(\frac{r}{r_0}\right)^{-2\gamma}$$

In general

$$F_m(\nu) = F_{m0}\left(\frac{r}{r_0}\right)^{-\frac{(7\gamma+3)}{(\gamma+4)}}$$

We also showed that

$$\Theta(t) = \Theta_0\left(\frac{r}{r_0}\right)$$

Van der Laan showed that a plasmoid expanding into a medium

$$\left(\frac{r}{r_0}\right) = \left(\frac{t}{\beta t_0}\right)^{\beta}$$

One can then show that a secular evolution of the flux density from an expanding source

$$F(\nu,t) = F_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{r}{r_0}\right)^{3}\left[\frac{1-e^{-\left[\tau_{\nu m}\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\gamma+4}{2}\right)}\left(\frac{r}{r_0}\right)^{-(7\gamma+3)}\right]}}{1-e^{-\tau_{\nu m}}}\right]$$

For $\beta \to 1$

$$F(\nu,t) = F_{m0}\left(\frac{\nu}{\nu_{m0}}\right)^{5/2}\left(\frac{t}{t_0}\right)^{3}\left[\frac{1-e^{-\left[\tau_{\nu m}\left(\frac{\nu}{\nu_{m0}}\right)^{-\left(\frac{\gamma+4}{2}\right)}\left(\frac{t}{t_0}\right)^{-(7\gamma+3)}\right]}}{1-e^{-\tau_{\nu m}}}\right]$$

where $F_{m0} = F(\nu_{m0})$ is the flux maximum at $t = t_0$

One can also show that the secular evolution of the flux takes on the form, using $F(\nu_m) = F_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)}$:

$$\dot{F}(\nu=\nu_m, r) = F_0(\nu=\nu_{m0})\frac{d}{dt}\left[\left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)}\right]$$

$$= -\frac{(7\gamma+3)}{(\gamma+4)} F_0(\nu=\nu_{m0})\left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)-\left(\frac{\gamma+4}{\gamma+4}\right)}$$

$$\times \frac{d}{dt}\left(\frac{r}{r_0}\right)$$

This results in:

$$\dot{F}(\nu=\nu_m, r) = F_0(\nu=\nu_{m0})\left[-\left(\frac{7\gamma+3}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{-\left(\frac{8\gamma+7}{\gamma+4}\right)}\left(\frac{\dot{r}}{r_0}\right)\right]$$

$$= -\left(\frac{7\gamma+3}{\gamma+4}\right) F_0(\nu=\nu_{m0})\left(\frac{r}{r_0}\right)^{-\left(\frac{8\gamma+7}{\gamma+4}\right)}\frac{\dot{r}}{r_0}$$

Then

$$\frac{\dot{F}(\nu=\nu_m, r)}{F(\nu=\nu_m, r)} = \frac{-\left(\frac{7\gamma+3}{\gamma+4}\right) F_0(\nu=\nu_{m0})\left(\frac{r}{r_0}\right)^{-\left(\frac{8\gamma+7}{\gamma+4}\right)}\frac{\dot{r}}{r_0}}{F_0(\nu=\nu_{m0})\left(\frac{r}{r_0}\right)^{-\left(\frac{7\gamma+3}{\gamma+4}\right)}}$$

$$= -\left(\frac{7\gamma+3}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{-\left(\frac{8\gamma+7}{\gamma+4}\right)}\left(\frac{r}{r_0}\right)^{\left(\frac{7\gamma+3}{\gamma+4}\right)}\frac{\dot{r}}{r_0}$$

$$= -\left(\frac{7\gamma+3}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{\frac{-8\gamma-7+7\gamma+3}{\gamma+4}}\left(\frac{\dot{r}}{r_0}\right)$$

$$= -\left(\frac{7\gamma+3}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{\underbrace{\frac{-\gamma-4}{\gamma+4}}_{-\left(\frac{\gamma+4}{\gamma+4}\right)}}\frac{\dot{r}}{r_0}$$

$$= -\left(\frac{7\gamma+3}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{-1}\frac{\dot{r}}{r_0}$$

This gives

$$\frac{\dot{F}(\nu=\nu_m, r)}{F(\nu=\nu_m, r)} = -\left(\frac{7\gamma+3}{\gamma+4}\right)\frac{\dot{r}}{r}$$

In similar fashion one can get an idea for the evolution of the maximum frequency. We showed earlier

$$\nu_m(r) = \nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)}$$

$$\dot{\nu}_m(r) = \nu_{m0}\frac{d}{dt}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)}$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)-\left(\frac{\gamma+4}{\gamma+4}\right)}\left(\frac{\dot{r}}{r_0}\right)$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\nu_{m0}\left(\frac{r}{r_0}\right)^{\frac{-4\gamma-6-\gamma-4}{\gamma+4}}\left(\frac{\dot{r}}{r_0}\right)$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{5\gamma+10}{\gamma+4}\right)}\frac{\dot{r}}{r_0}$$

Then

$$\frac{\dot{\nu}_m(r)}{\nu_m(r)} = \frac{-\left(\frac{4\gamma+6}{\gamma+4}\right)\nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{5\gamma+10}{\gamma+4}\right)}\frac{\dot{r}}{r_0}}{\nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)}}$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{\frac{-(5\gamma+10)+(4\gamma+6)}{\gamma+4}}\frac{\dot{r}}{r_0}$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{\frac{-\gamma-4}{\gamma+4}}\left(\frac{\dot{r}}{r_0}\right)$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{-\left(\frac{\gamma+4}{\gamma+4}\right)}\frac{\dot{r}}{r_0}$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\left(\frac{r}{r_0}\right)^{-1}\frac{\dot{r}}{r_0}$$

$$= -\left(\frac{4\gamma+6}{\gamma+4}\right)\frac{\dot{r}}{r}$$

Then one can show that

$$\nu \gg \nu_m : \quad S(\nu,r) = S_0(\nu)\left(\frac{r}{r_0}\right)^{-2\gamma}$$

$$\dot{S}(\nu,r) = S_0(\nu)\left[-2\gamma\left(\frac{r}{r_0}\right)^{-2\gamma-1}\frac{\dot{r}}{r_0}\right]$$

This results in

$$\dot{S}(\nu,r) = S_0(\nu)\left[-2\gamma\left(\frac{r}{r_0}\right)^{-2\gamma}\left(\frac{r}{r_0}\right)^{-1}\left(\frac{\dot{r}}{r_0}\right)\right]$$

$$= S_0(\nu)\left[-2\gamma\left(\frac{r}{r_0}\right)^{-2\gamma}\frac{\dot{r}}{r}\right]$$

$$\frac{\dot{S}(\nu,r)}{S(\nu,r)} = \frac{S_0(\nu)\left[-2\gamma\left(\frac{r}{r_0}\right)^{-2\gamma}\frac{\dot{r}}{r}\right]}{S_0(\nu)\left(\frac{r}{r_0}\right)^{-2\gamma}}$$

$$= -2\gamma\left(\frac{\dot{r}}{r}\right)$$

For $\nu << \nu_m$: $F(\nu,r) = F_0(\nu)\left(\frac{r}{r_0}\right)^3$

$$\dot{F}(\nu,r) = F_0(\nu)\frac{d}{dt}\left(\frac{r}{r_0}\right)^3$$

$$= 3F_0(\nu)\left(\frac{r}{r_0}\right)^2\frac{\dot{r}}{r_0}$$

$$\frac{\dot{F}(\nu,r)}{F(\nu,r)} = \frac{3F_0(\nu)\left(\frac{r}{r_0}\right)^2\left(\frac{\dot{r}}{r_0}\right)}{F_0(\nu)\left(\frac{r}{r_0}\right)^3}$$

$$= 3\left(\frac{r}{r_0}\right)^{-1}\left(\frac{\dot{r}}{r_0}\right) = 3\left(\frac{\dot{r}}{r}\right)$$

Long term spectral behaviour

For a constant spectral index γ the integrated energy contained in a given synchrotron spectrum depends on F_{m0} for all the flaring events. If we assume that the occurrence of flaring events with initial $\left(\left(\frac{r}{r_0}\right) \sim 1\right)$ flux is such that the higher flux density flares are occurring less often, then

Distribution function of flares. This is determined through observations.

$$f(F(\nu)) \propto F(\nu)^{-\epsilon}$$ [number / flux density]

This flaring flux density distribution function will allow us to determine the time-averaged flux density at all frequencies where the flux density peaked for the different flares:

$$\Rightarrow \langle F(\nu, r) \rangle \propto \int F(\nu, r)\, f(F(\nu))\, dF(\nu)$$

For relativistic electron clouds expanding into a thermal plasma of optical depth $\tau_\nu \sim 1$ the spectrum will be mostly self-absorbed.

We showed earlier that the flux density

$$F(\nu) \propto \nu^{5/2} \quad \text{[self-absorbed spectrum]}$$

$$f(F(\nu)) \propto F(\nu)^{-\varepsilon}$$

$$\propto \left(\nu^{5/2}\right)^{-\varepsilon}$$

$$\propto \nu^{-\frac{5\varepsilon}{2}}$$

Also

$$\frac{dF}{d\nu} \propto 5/2 \, \nu^{3/2}$$

$$dF(\nu) \propto \nu^{3/2} \, d\nu$$

Therefore

$$\langle F(\nu, r) \rangle \propto \int F(\nu, r) \, f(F(\nu)) \, dF(\nu)$$

$$\propto \int \nu^{5/2} \left(\nu^{5/2}\right)^{-\varepsilon} \nu^{3/2} \, d\nu$$

This gives

$$\langle F(\nu,r) \rangle \propto \int \nu^{5/2 - \frac{5\varepsilon}{2} + 3/2} \, d\nu$$

$$\propto \int \nu^{\frac{8 - 5\varepsilon}{2}} \, d\nu$$

$$\propto \frac{\nu^{\frac{8-5\varepsilon}{2} + \frac{2}{2}}}{\frac{8-5\varepsilon}{2} + \frac{2}{2}}$$

$$\propto \nu^{\frac{10 - 5\varepsilon}{2}}$$

$$\propto \nu^{5/2(2 - \varepsilon)}$$

If one observe a particular source the observations will tell us what ε is. For a rapidly flaring source AG Aqr $\varepsilon \approx 1.8$, which gives

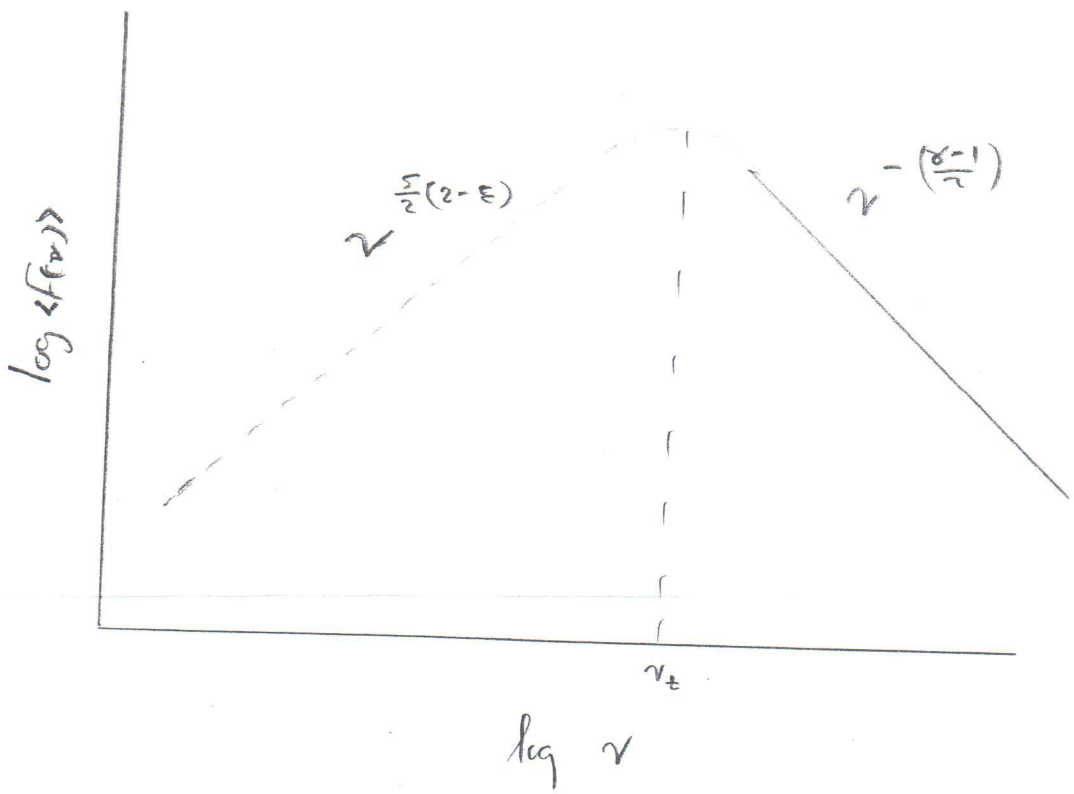
$$\langle F(\nu,r) \rangle \propto \nu^{5/2(2-1.8)}$$

$$\propto \nu^{0.5}$$

One typically observe that the time-averaged spectrum

$$\langle F(\nu,r) \rangle \propto \nu^{0.5}$$

show an absorbed spectral profile up to a certain frequency where all the blobs will be optically thin to the internal synchrotron emission.



Here ν_t represents a turn-over frequency where all the blobs become optically thin to the emission