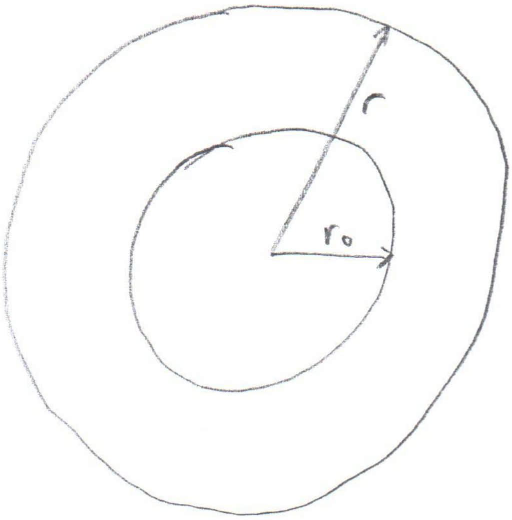
Synchrotron Flares from expanding electron clouds and jets

Van der Laan Model

This was one of the early models to explain radio synchrotron emission from expanding clouds of relativistic electrons imbedded in a thermal plasma. Consider a spherical cloud of relativistic electrons with spectrum

$$N(E) = \overset{[cm^{-3}\ erg^{\gamma+1}]}{K}\ E^{-\gamma} \quad [cm^{-3}\ erg^{-1}]$$

Let us also assume that the cloud contains magnetic flux that is frozen-in and that the magnetic flux is isotropic. In other words the magnetic field is chaotic inside the cloud. Let us now assume that this cloud is expanding adiabatically. Since magnetic flux will be conserved we will have the following:



Consider a sphere with radius r_0 at t_0. Its surface area is $A_0 = 4\pi r_0^2$. At a time $t > t_0$ its surface area is $A(t) = 4\pi r(t)^2$. The ratio of the initial area at t_0 and the area at time t is:

$$\frac{A(t)}{A_0} = \frac{4\pi r(t)^2}{4\pi r_0^2} = \left(\frac{r(t)}{r_0}\right)^2$$

Therefore

$$A(t) = A_0 \left(\frac{r(t)}{r_0}\right)^2$$

Since magnetic flux is conserved

$$\phi_B = \int \bar{B} \cdot d\bar{A} = \text{const}$$

$$B_0 A_0 = B(t) A(t)$$

$$\Rightarrow \quad B(t) = B_0 \left[\frac{A_0}{A(t)}\right]$$

$$= B_0 \left[\frac{r_0}{r(t)}\right]^2$$

$$= B_0 \left[\frac{r(t)}{r_0}\right]^{-2}$$

$$= B_0 \left(\frac{r}{r_0}\right)^{-2}$$

As a result of the expanding the cloud of relativistic electrons it will cool. We can explain this through standard thermodynamics. From the first law of thermodynamics we have

$$\Delta U = \Delta Q - P \Delta V$$

For an adiabatic process $\Delta Q = 0$

$$dE = -P\, dV$$

The eqn of state of a relativistic electron gas is

$$P = \frac{1}{3}\left(\frac{E}{V}\right)$$

$$\Rightarrow \quad dE = -P\, dV$$
$$= -\frac{1}{3}\frac{E}{V}\, dV$$

$$\Rightarrow \quad \frac{dE}{E} = -\frac{1}{3}\frac{dV}{V}$$

$$\Rightarrow \quad \int_{E_0}^{E} \frac{dE}{E} = -\frac{1}{3}\int_{V_0}^{V} \frac{dV}{V}$$

$$\left[\ln E\right]_{E_0}^{E} = -\frac{1}{3}\left[\ln V\right]_{V_0}^{V}$$

$$\ln\left(\frac{E}{E_0}\right) = -\frac{1}{3}\ln\left(\frac{V}{V_0}\right)$$

$$= \ln\left(\frac{V}{V_0}\right)^{-1/3}$$

$$\left(\frac{E}{E_0}\right) = \left(\frac{V}{V_0}\right)^{-1/3}$$

$$= \left(\frac{\frac{4\pi}{3}r^3}{\frac{4\pi}{3}r_0^3}\right)^{-1/3}$$

{Here we assume a spherical expansion of the electron cloud.}

$$= \left[\left(\frac{r}{r_0}\right)^3\right]^{-1/3}$$

$$= \left(\frac{r}{r_0}\right)^{-1}$$

$$\Rightarrow \quad E = E_0\left(\frac{r}{r_0}\right)^{-1}$$

This implies that every particle that participates in the expansion loss energy according to $(r/r_0)^{-1}$. This adiabatic energy loss will not change the shape of the spectrum but it only changes the amplitude. If the number of particles inside the expanding cloud is conserved we have the following :

$$\frac{d}{dt}\left\{ r^3(t) \int_{E_1}^{E_2} K(t,r)\, E^{-\gamma}\, dE \right\} = 0$$

This implies that

$$r^3(t) \int_{E_1}^{E_2} K(t,r) E^{-\gamma}\, dE = \text{Const}$$

$$\Rightarrow \quad r^3(t) \int_{E_1}^{E_2} K(t,r) E^{-\gamma}\, dE = r_0^3 \int_{E_{01}}^{E_{02}} K(t_0, r_0)\, E^{-\gamma}\, dE$$

$$\Rightarrow \quad r^3(t)\, K(t,r) \int_{E_1}^{E_2} E^{-\gamma}\, dE = r_0^3\, K(t_0, r_0) \int_{E_{01}}^{E_{02}} E^{-\gamma}\, dE$$

Notice that E_{01} and E_{02} are the values of E_1 and E_2 at r_0 respectively. The value of the integral is :

$$\int_{E_1}^{E_2} E^{-\gamma}\, dE = \left[\frac{E^{-\gamma+1}}{-\gamma+1}\right]_{E_1}^{E_2}$$

$$= \left[\frac{E_2^{-\gamma+1}}{-\gamma+1}\right] - \left[\frac{E_1^{-\gamma+1}}{-\gamma+1}\right]$$

$$= \frac{E_2^{-\gamma+1} - E_1^{-\gamma+1}}{1-\gamma}$$

$$= \frac{E_1^{-\gamma+1} - E_2^{-\gamma+1}}{\gamma - 1}$$

This can now be written as follows

$$r^3(t)\, K(t,r) \left[\frac{E_1^{-\gamma+1} - E_2^{-\gamma+1}}{\gamma - 1} \right] = r_0^3\, K(t_0, r_0) \left[\frac{E_{01}^{-\gamma+1} - E_{02}^{-\gamma+1}}{\gamma - 1} \right]$$

Let's write this as follows:

$$r_0^3 K_0 \left[\frac{E_{01}^{-\gamma+1} - E_{02}^{-\gamma+1}}{\gamma - 1} \right] = r^3 K \left[\frac{E_1^{-\gamma+1} - E_2^{-\gamma+1}}{\gamma - 1} \right]$$

Using this relation we can determine the r-dependence of K. This is done as follows:

Using the r-dependence of energy

$$\Rightarrow \quad E = E_0 \left(\frac{r}{r_0}\right)^{-1}$$

we can develop the equation for the conservation of particles

$$r_0^3 K_0 \left[\frac{E_{01}^{-\gamma+1} - E_{02}^{-\gamma+1}}{\gamma - 1} \right] = r^3 K \left[\frac{\left[E_{01}\left(\frac{r}{r_0}\right)^{-1}\right]^{-\gamma+1} - \left[E_{02}\left(\frac{r}{r_0}\right)^{-1}\right]^{-\gamma+1}}{\gamma - 1} \right]$$

$$r_0^3 K_0 \left[E_{01}^{-\gamma+1} - E_{02}^{-\gamma+1} \right] = r^3 K \left(\frac{r}{r_0}\right)^{\gamma-1} \left[E_{01}^{-\gamma+1} - E_{02}^{-\gamma+1} \right]$$

The energy term cancels out leaving:

$$r_0^3 K_0 = r^3 K \left(\frac{r}{r_0}\right)^{\gamma-1}$$

$$\Rightarrow \quad K = K_0 \left(\frac{r_0}{r}\right)^3 \left(\frac{r}{r_0}\right)^{-(\gamma-1)}$$

$$= K_0 \left(\frac{r}{r_0}\right)^{-3} \left(\frac{r}{r_0}\right)^{-(\gamma-1)}$$

$$= K_0 \left(\frac{r}{r_0}\right)^{-\gamma+1-3}$$

$$= K_0 \left(\frac{r}{r_0}\right)^{-\gamma-2}$$

$$= K_0 \left(\frac{r}{r_0}\right)^{-(\gamma+2)}$$

Therefore the particle spectrum can be written as follows

$$N(E) = K\, E^{-\gamma}$$

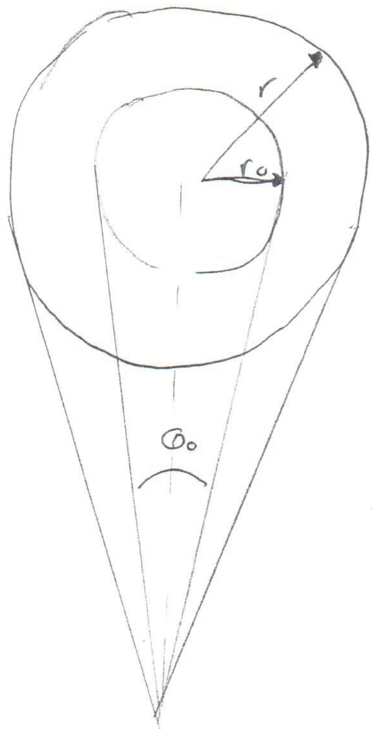
$$= K_0 \left(\frac{r}{r_0}\right)^{-(\gamma+2)} \left[E_0 \left(\frac{r}{r_0}\right)^{-1}\right]^{-\gamma}$$

$$= K_0 E_0^{-\gamma} \left(\frac{r}{r_0}\right)^{-(\gamma+2)} \left(\frac{r}{r_0}\right)^{\gamma}$$

$$= K_0 E_0^{-\gamma} \left(\frac{r}{r_0}\right)^{-2}$$

↑ The particle spectrum is also position dependent.

The dependence of the angular size of the source with blob size also gives us the following relationship.



$$\theta_0 = \frac{2r_0}{D}$$

$$\theta = \frac{2r}{D}$$

$$\Rightarrow \frac{\theta}{\theta_0} = \frac{\left(\frac{2r}{D}\right)}{\left(\frac{2r_0}{D}\right)}$$

$$= \left(\frac{r}{r_0}\right)$$

$$\Rightarrow \theta = \theta_0 \left(\frac{r}{r_0}\right)$$

Now that we have defined the particle density, magnetic field and energy in terms of the position, we can now proceed to calculate the synchrotron flux density as a function of position.

The synchrotron flux can be written as follows:

$$F_s(\nu) = 4\pi\, \mathcal{E}_s(\nu)\, \frac{V}{4\pi D^2} \qquad [\mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}}]$$

$$\mathcal{E}_s(\nu) \propto K\, B^{\left(\frac{p+1}{2}\right)}\, \nu^{-\left(\frac{\gamma-1}{2}\right)} \qquad [\mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}\ sr^{-1}}]$$

Therefore

$$F_s(\nu) \propto 4\pi\, K\, B^{\left(\frac{\gamma+1}{2}\right)}\, \nu^{-\left(\frac{\gamma-1}{2}\right)}\, \frac{V}{4\pi D^2}$$

$$\propto \frac{K r^3}{D^2}\, B^{\left(\frac{\gamma+1}{2}\right)}\, \nu^{-\left(\frac{\gamma-1}{2}\right)}$$

$$\propto \mathrm{Const}\ K \left(\frac{r}{D}\right)^3 B^{\left(\frac{\gamma+1}{2}\right)}\, \nu^{-\left(\frac{\gamma-1}{2}\right)}$$

$$\propto \mathrm{Const}\ K\, \underset{\uparrow}{\Theta}^3\, B^{\left(\frac{\gamma+1}{2}\right)}\, \nu^{-\left(\frac{\gamma-1}{2}\right)}$$

angular size of source

Since K, Θ, B have all angular dependence we can write

$$F_s(\nu) = \mathrm{Const}\ K(r)\, \Theta(r)^3\, B(r)^{\left(\frac{\gamma+1}{2}\right)}\, \nu^{-\left(\frac{\gamma-1}{2}\right)}$$

$$F_s(\nu) = \text{Const}\left[K_0\left(\frac{r}{r_0}\right)^{-\gamma-2}\right]\left[\Theta_0\left(\frac{r}{r_0}\right)\right]^3\left[B_0\left(\frac{r}{r_0}\right)^{-2}\right]^{\left(\frac{\gamma+1}{2}\right)}\nu^{-\left(\frac{\gamma-1}{2}\right)}$$

$$= \text{Const}\, K_0\,\Theta_0^3\, B_0^{\frac{\gamma+1}{2}}\left[\left(\frac{r}{r_0}\right)^{-\gamma-2}\left(\frac{r}{r_0}\right)^3\left(\frac{r}{r_0}\right)^{-(\gamma+1)}\right]\nu^{-\left(\frac{\gamma-1}{2}\right)}$$

$$= \text{Const}\, K_0\,\Theta_0^3\, B_0^{\frac{\gamma+1}{2}}\,\nu^{-\left(\frac{\gamma-1}{2}\right)}\left(\frac{r}{r_0}\right)^{-\gamma-2+3-\gamma-1}$$

$$= \text{Const}\, K_0\,\Theta_0^3\, B_0^{\frac{\gamma+1}{2}}\,\nu^{-\left(\frac{\gamma-1}{2}\right)}\left(\frac{r}{r_0}\right)^{-2\gamma}$$

$$= F_0(\nu)\left(\frac{r}{r_0}\right)^{-2\gamma}\left(\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}\right)$$

Notice that

$$F_0(\nu) = \text{Const}\, K_0\,\Theta_0^3\, B_0^{\frac{\gamma+1}{2}}\,\nu^{-\left(\frac{\gamma-1}{2}\right)}$$

We can now use this equation to determine the frequency where the maximum emission occurs in terms of position. This is determined as follows:

Consider a plasma cloud with a magnetic field $\bar{H}$. The presence of the magnetic field results in the emission of synchrotron radiation. If we assume that the bulk of the synchrotron emission occurs at the frequency $\nu = \nu_c$, with

$$\nu_c = \frac{3}{4\pi} \gamma^3 \omega_B \sin\theta \qquad \omega = 2\pi\nu$$

$$= \frac{3}{4\pi} \gamma^3 \frac{qB}{\gamma m_e c} \sin\theta$$

$$= \frac{3}{4\pi} \gamma^2 \frac{qB}{m_e c} \sin\theta$$

This allows us to solve for

$$\gamma^2 = \frac{4\pi}{3} \left(\frac{m_e c}{qB}\right) \sin^{-1}\theta \; \nu_c$$

$$\gamma = \sqrt{\frac{4\pi}{3}} \left(\frac{m_e c}{qB}\right)^{-1/2} \sin^{-1/2}\theta \; \nu_c^{1/2}$$

For synchrotron emission the relation between the emitted frequency and the Larmor frequency of the electrons is

$$\nu_s \approx \gamma^2 \nu_L$$

$$\Rightarrow \gamma^2 = \left(\frac{\nu_s}{\nu_L}\right)$$

$$\nu_L = \frac{qB}{2\pi m_e c}$$

$$\Rightarrow \gamma = \left(\frac{\nu_s}{\nu_L}\right)^{1/2}$$

We assume that at a particular frequency ν the emission is absorbed by those electrons that can emit it. The kinetic temperature of the electrons

$$kT_e = \gamma m_e c^2$$

$$= \left(\frac{\nu_s}{\nu_L}\right)^{1/2} m_e c^2$$

For an absorbed source the brightness temperature can be obtained from the Rayleigh-Jeans law

$$S(\nu) = 2kT_B \frac{\nu^2}{c^2} \quad (\mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}\ sr^{-1}})$$

For a compact source spanning a solid angle $\Delta\Omega \sim \theta_s^2$ we have

$$F(\nu) = S(\nu)\,\Delta\Omega_s$$

$$\approx S(\nu)\,\theta_s^2 \qquad \theta_s = \left(\frac{d}{D}\right)$$

$$\approx 2kT_B\left(\frac{\nu}{c}\right)^2\theta_s^2 \qquad [\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}}]$$

Solving for kT_B:

$$kT_B = \frac{1}{2}F(\nu)\left(\frac{\nu}{c}\right)^{-2}\theta_s^{-2}$$

We showed earlier

$$kT_e = \left(\frac{\nu}{\nu_L}\right)^{1/2} m_e c^2$$

Then equating the kinetic and brightness temperatures we can solve for ν_m, where the spectrum peaks.

$$\Rightarrow \quad kT_e = kT_B$$

$$\left(\frac{\nu_s}{\nu_L}\right)^{1/2} m_e c^2 = \frac{F(\nu_s)}{2\left(\frac{\nu_s}{c}\right)^2\theta_s^2}$$

$$\frac{\nu_s^{5/2}}{\nu_L^{1/2}}\,m_e c^2 = \frac{F(\nu_s)\,c^2}{2\,\theta_s^2}$$

$$\nu_s^{5/2} = \frac{1}{2}\,\theta_s^{2}\,\nu_L^{1/2}\,F(\nu_s)$$

Then

$$\nu_s^{5/2} = \frac{1}{2}\left(\frac{qB}{2\pi m_e c}\right)^{1/2} \theta_s^{-2} \; F(\nu_s)$$

$$= \text{Const} \; B^{1/2} \theta_s^{-2} F(\nu_s)$$

Now for $F(\nu)$ we can use

$$F(\nu_m) = F(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\alpha}$$ ← will justify this choice

where ν_t is any frequency where the source is optically thin. Then

$$\nu_m^{5/2} = \text{Const} \; B^{1/2} \theta_s^{-2} \; F(\nu_m)$$

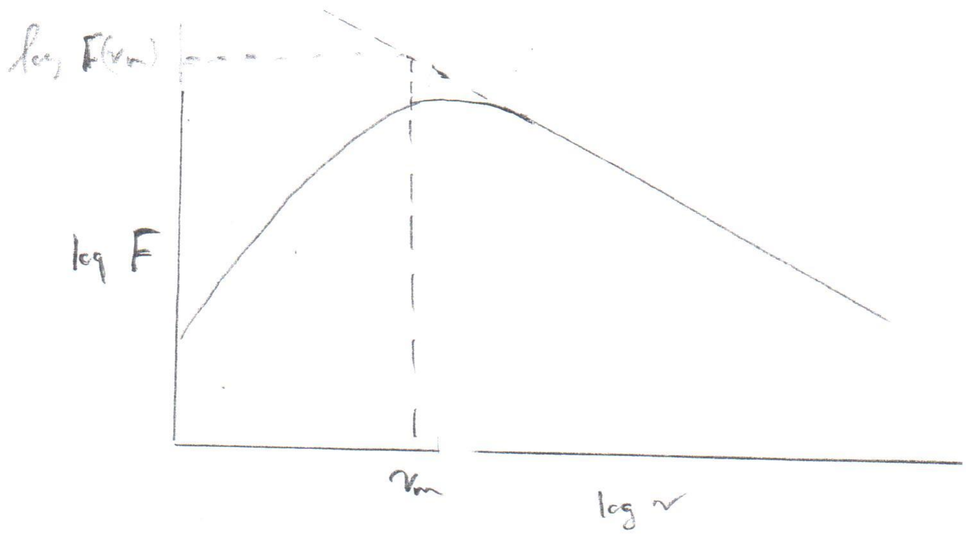
$$= \text{Const} \; B^{1/2} \theta_s^{-2} \left[F(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\alpha}\right]$$

$$\nu_m = \text{Const} \; B^{1/5} \theta_s^{-4/5} \left[F(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\alpha}\right]^{2/5}$$

$$= \text{Const} \; B^{1/5} \theta_s^{-4/5} \left[F(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{p-1}{2}\right)}\right]^{2/5}$$

We need to quantify something first:

For an optically thick source the brightness can be written as



The flux density in the optically thin part is

$$F(\nu_t) = F(\nu_m)\left(\frac{\nu_t}{\nu_m}\right)^{-\alpha} \qquad \alpha = \left(\frac{\gamma-1}{2}\right)$$

$$\Rightarrow \quad F(\nu_m) = F(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\alpha}$$

Therefore

$$\nu_m = \text{Const}\; B^{1/5}\, \theta_s^{-4/5} \left[F(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{2}\right)}\right]^{2/5}$$

Now we have to remember that in the expression for flux density of the expanding source the flux $F(\nu_s)$, θ and B have radial dependences, i.e

- $$F_s(\nu) = F_0(\nu)\left(\frac{r}{r_0}\right)^{-2\gamma}$$
- $$\theta = \theta_0\left(\frac{r}{r_0}\right)$$
- $$B = B_0\left(\frac{r}{r_0}\right)^{-2}$$

This can now be substituted into the equation for ν_m

$$\nu_m = \text{Const } B^{1/5}\,\theta_s^{-4/5}\left[F(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{2}\right)}\right]^{2/5}$$

$$= \text{Const}\left[B_0\left(\frac{r}{r_0}\right)^{-2}\right]^{1/5}\left[\theta_0\left(\frac{r}{r_0}\right)\right]^{-4/5} \times$$
$$\left[F_0(\nu_t)\left(\frac{r}{r_0}\right)^{-2\gamma}\left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{2}\right)}\right]^{2/5}$$

$$= \text{Const } B_0^{1/5}\,\theta_0^{-4/5}\,F_0(\nu_t)^{2/5}\left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{5}\right)} \times$$
$$\left(\frac{r}{r_0}\right)^{-2/5}\left(\frac{r}{r_0}\right)^{-4/5}\left(\frac{r}{r_0}\right)^{-\frac{4\gamma}{5}}$$

$$= \text{Const } B_0^{1/5}\,\theta_0^{-4/5}\,F_0(\nu_t)^{2/5}\left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{5}\right)}\left(\frac{r}{r_0}\right)^{-2/5-\frac{4}{5}-\frac{4\gamma}{5}}$$

$$= \text{Const } B_0^{1/5}\,\theta_0^{-4/5}\,F_0^{2/5}(\nu_t)\left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{5}\right)}\left(\frac{r}{r_0}\right)^{\frac{-(4\gamma+6)}{5}}$$

If $\left(\frac{r}{r_0}\right) \approx 1$

Needs to be modified to express it into ν_{mo}

$$\nu_m = \nu_{mo}$$

$$= \mathrm{Const}\; B_0^{1/5}\, \Theta_0^{-4/5}\, F_0^{2/5}(\nu_t) \left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{5}\right)}$$

Then we can cast the frequency where the spectrum peaks in the following way

$$\nu_m = \mathrm{Const}\; B_0^{1/5}\, \Theta_0^{-4/5}\, F_0^{2/5}(\nu_t) \left(\frac{\nu_t}{\nu_m}\right)^{\left(\frac{\gamma-1}{5}\right)} \left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{5}\right)}$$

$$\nu_m\, \nu_m^{\frac{\gamma-1}{5}} = \mathrm{Const}\; B_0^{1/5}\, \Theta_0^{-4/5}\, F_0^{2/5}(\nu_t)\, \nu_t^{\frac{\gamma-1}{5}}\, \nu_{mo}^{\left(\frac{\gamma-1}{5}\right)}\, \nu_{mo}^{-\left(\frac{\gamma-1}{5}\right)}$$

$$\times \left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{5}\right)}$$

$$\nu_m^{\frac{\gamma+4}{5}} = \mathrm{Const}\; B_0^{1/5}\, \Theta_0^{-4/5}\, F_0^{2/5}(\nu_t) \left(\frac{\nu_t}{\nu_{mo}}\right)^{\frac{\gamma-1}{5}} \nu_{mo}^{\left(\frac{\gamma-1}{5}\right)}$$

$$\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{5}\right)}$$

$$= \mathrm{Const}\; B_0^{1/5}\, \Theta_0^{-4/5}\, F_0^{2/5}(\nu_t) \left(\frac{\nu_t}{\nu_{mo}}\right)^{\left(\frac{\gamma-1}{5}\right)} \times$$

$$\left[\mathrm{Const}\; B_0^{1/5}\, \Theta_0^{-4/5}\, F_0^{2/5}(\nu_t) \left(\frac{\nu_t}{\nu_{mo}}\right)^{\frac{\gamma-1}{5}}\right]^{\frac{\gamma-1}{5}}$$

$$\times \left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{5}\right)}$$

$$\nu_m^{\frac{\delta+4}{5}} = \text{Const}\left[B_0^{1/5}\left(B_0^{1/5}\right)^{\frac{\delta-1}{5}}\right]\left[\Theta_0^{-4/5}\left(\Theta_0^{-4/5}\right)^{\frac{\delta-1}{5}}\right]$$

$$\left[F_0(\nu_t)^{2/5}\left(F_0(\nu_t)^{2/5}\right)^{\frac{\delta-1}{5}}\right]\left[\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{\delta-1}{5}}\left[\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{\delta-1}{5}}\right]^{\frac{\delta-1}{5}}\right]$$

$$\times\left(\frac{r}{r_0}\right)^{-\left(\frac{4\delta+6}{5}\right)}$$

This results in

$$\nu_m^{\frac{\delta+4}{5}} = \text{Const}\; B_0^{1/5+\frac{\delta-1}{25}}\;\Theta_0^{-4/5-\frac{4}{5}\left(\frac{\delta-1}{5}\right)}\;F_0(\nu_t)^{2/5+\frac{2(\delta-1)}{25}}$$

$$\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{\delta-1}{5}+\frac{(\delta-1)^2}{25}}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\delta+6}{5}\right)}$$

$$= \text{Const}\; B_0^{\frac{\delta+4}{25}}\;\Theta_0^{-\frac{20-4(\delta-1)}{25}}\;F_0(\nu_t)^{\frac{10+2(\delta-1)}{25}}$$

$$\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{5(\delta-1)+(\delta-1)^2}{25}}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\delta+6}{5}\right)}$$

$$= \text{Const}\; B_0^{\frac{\delta+4}{25}}\;\Theta_0^{-\frac{(4\delta+16)}{25}}\;F_0(\nu_t)^{\frac{2\delta+8}{25}}\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{(\delta-1)\lbrace 5+\delta-}{25}}$$

$$\left(\frac{r}{r_0}\right)^{-\left(\frac{4\delta+6}{5}\right)}$$

This results in

$$\nu_m^{\frac{\gamma+4}{5}} = \text{Const}\; B_0^{\frac{\gamma+6}{25}}\; \Theta_0^{-\frac{6(\gamma+4)}{25}}\; F_0(\nu_t)^{\frac{2(\gamma+4)}{25}}$$

$$\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{(\gamma-1)(\gamma+4)}{25}} \times \left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{5}\right)}$$

Therefore

$$\nu_m = \text{Const}\left(B_0^{\frac{\gamma+6}{25}}\right)^{\frac{5}{\gamma+4}}\left(\Theta_0^{-\frac{6(\gamma+4)}{25}}\right)^{\frac{5}{\gamma+4}}$$

$$\left(F_0(\nu_t)^{\frac{2(\gamma+4)}{25}}\right)^{\frac{5}{\gamma+4}}\left(\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{(\gamma-1)(\gamma+4)}{25}}\right)^{\frac{5}{\gamma+4}}$$

$$\times\left(\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{5}\right)}\right)^{\frac{5}{\gamma+4}}$$

$$= \text{Const}\; B_0^{1/5}\; \Theta_0^{-4/5}\; F_0(\nu_t)^{2/5}\left(\frac{\nu_t}{\nu_{m0}}\right)^{\frac{\gamma-1}{5}} \times$$

$$\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)}$$

$$= \nu_{m0}\left(\frac{r}{r_0}\right)^{-\left(\frac{4\gamma+6}{\gamma+4}\right)}$$

where

$$\nu_{m0} = \text{Const}\; B_0^{1/5}\; \Theta_0^{-4/5}\; F_0(\nu_t)^{2/5}\left(\frac{\nu_t}{\nu_{m0}}\right)^{\left(\frac{\gamma-1}{5}\right)}$$

The maximum radio emission can now be determined from an expanding synchrotron emitting cloud of electrons. We showed earlier that

$$F_s(\nu) = \text{Const}\ \underset{\uparrow\ \text{From } e^- \text{ spectrum}}{K}\ \theta_s^3\ B^{\frac{\delta+1}{2}}\ \nu^{-\left(\frac{\delta-1}{2}\right)}$$

All the quantities K, θ_s, B_s and ν depend on the source size through

- $K = K_0 \left(\frac{r}{r_0}\right)^{-(\delta+2)}$
- $\theta = \theta_0 \left(\frac{r}{r_0}\right)$
- $B = B_0 \left(\frac{r}{r_0}\right)^{-2}$
- $\nu_m = \nu_{m0} \left(\frac{r}{r_0}\right)^{-\left(\frac{4\delta+6}{\delta+4}\right)}$

Substituting these into the expression for the flux density one get the following expression