The Lorentz Transformations

We know that the linking between coördinates of events in different reference frames are given by the Lorentz transformations

$$x' = \Gamma(x - Vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \Gamma\left(t - \frac{Vx}{c^2}\right)$$

and inverse transformations

$$x = \Gamma(x' + Vt')$$
$$y = y'$$
$$z = z'$$
$$t = \Gamma\left(t' + \frac{Vx'}{c^2}\right)$$

The transformations and inverse transformations gives us a tool of transforming the coördinates of vectors between reference frames. Now if

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

we can from the Lorentz transformations get

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \begin{bmatrix} \Gamma & -\beta\Gamma & 0 & 0 \\ -\beta\Gamma & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

This gives

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} \, x^{\nu}$$

Similarly the inverse transformations give

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \Gamma & \beta\Gamma & 0 & 0 \\ \beta\Gamma & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix}$$

$$x^{\mu} = \Lambda^{\mu}{}_{\nu'} \, x^{\nu'}$$

Here we have the two transformation matrices to transform the coordinates for at least contravariant vectors from one reference frame to another which is in a state of constant motion with respect to each other

$$\Rightarrow \quad \Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \Gamma & -\beta\Gamma & 0 & 0 \\ -\beta\Gamma & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda^{\mu}{}_{\nu'} = \begin{bmatrix} \Gamma & \beta\Gamma & 0 & 0 \\ \beta\Gamma & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are another vector base, namely covariant vectors, which transforms like

$$u_{\mu'} = \Lambda^{\rho}{}_{\mu'} \, u_{\rho}$$

$$u_{\mu} = \Lambda^{\rho'}{}_{\mu} \, u_{\rho'}$$

With these the components of four-vectors can be transformed to different reference frames.

Four-vectors

Some important 4-vectors

4-velocity

$$U^\mu = \frac{dx^\mu}{d\tau} \qquad\qquad d\tau = dt\,(1-\beta^2)^{1/2}$$

$$= \left[\frac{dx^0}{d\tau},\ \frac{dx^i}{d\tau}\right]$$

$$\frac{dx^0}{d\tau} = \frac{d(ct)}{d\tau} = c\,\frac{dt}{d\tau}$$

What is $d\tau$

$$ds^2 = c^2 d\tau^2 = (dx^0)^2 - \sum_{i=1}^{3} (dx^i)^2$$

$$= c^2 dt^2 - \sum_{i=1}^{3} (dx^i)^2$$

$$= c^2 dt^2 \left[1 - \sum_{i=1}^{3} \frac{1}{c^2}\left(\frac{dx^i}{dt}\right)^2\right]$$

$$= c^2 dt^2 \left[1 - \frac{V^2}{c^2}\right]$$

$$d\tau = dt\left[1 - \frac{V^2}{c^2}\right]^{1/2}$$

$$dt = \frac{d\tau}{\left[1-\left(\frac{V}{c}\right)^2\right]^{1/2}}$$

$$= \Gamma\, d\tau$$

$\tau \Rightarrow t'$, i.e. the time measured on a stationary clock in moving frame k'

This im

$$\frac{dx^0}{d\tau} = c\,\frac{dt}{d\tau}$$

$$= \frac{c}{(1-\beta^2)^{1/2}}$$

$$= \Gamma c$$

Spatial components:

$$\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\,\frac{dt}{d\tau}$$

$$= \Gamma v^i$$

$$U^\mu = [\Gamma c ; \Gamma v^i]$$

The four-momentum

The four-momentum is defined by

$$p^\mu = m_0 U^\mu$$

$$= m_0 [\Gamma c ; \Gamma v^i]$$

$$= [\Gamma m_0 c ; \Gamma m_0 v^i]$$

$$= \left[\frac{\Gamma m_0 c^2}{c} ; \underbrace{\Gamma m_0}_{\text{Relativistic mass.}} v^i\right]$$

$$= \left[\frac{E}{c} ; p^i\right]$$

$$\Rightarrow \quad E = \Gamma m_0 c^2 = (p^2c^2 + m_0^2c^4)^{1/2}$$

The Invariance of Power

The zeroth component of the four-momentum relates to energy. To link it to power (rate of energy loss) we consider the four-force

$$F^{\mu} = \frac{dp^{\mu}}{d\tau}$$

The transformation for this is

$$A^{\mu} = \Lambda^{\mu}{}_{\nu'} A^{\nu'}$$

$$\Rightarrow \quad \frac{dp^{\mu}}{d\tau} = \Lambda^{\mu}{}_{\nu'} \frac{dp^{\nu'}}{d\tau}$$

The zeroth-component

$$\frac{dp^{0}}{d\tau} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dp^{0'}}{d\tau} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In k' we take the radiation to be isotropic which means that the spatial components of the radiation momentum cancels.

$$\frac{dp^{0}}{d\tau} = \gamma \frac{dp^{0'}}{d\tau} \qquad\qquad d\tau = \frac{dt}{\gamma}$$

$$\frac{d\left(\frac{E}{c}\right)}{\left(\frac{dt}{\gamma}\right)} = \frac{\gamma\, d\left(\frac{E'}{c}\right)}{\left(\frac{dt}{\gamma}\right)}$$

$$\gamma \frac{dE}{dt} = \gamma^{2} \frac{dE'}{dt}$$

This gives

$$\frac{dG}{dt} = \Gamma \frac{dG'}{dt}$$

$$= \frac{dG'}{\left(\frac{dt}{\Gamma}\right)}$$

$$= \frac{dG'}{d\tau}$$

$$= \frac{dG'}{dt'} \qquad dt' = d\tau$$

↑ dt' measures the time on a stationary clock in a moving reference frame

Hence

$$\left.\begin{aligned} P &= \frac{dG}{dt} \\ P' &= \frac{dE'}{dt'} \end{aligned}\right\} \quad \underset{[k]}{P} = \underset{[k']}{P'} \Rightarrow \underset{[k]}{\frac{dE}{dt}} = \underset{[k']}{\frac{dE'}{dt'}}$$

Power is an invariant, i.e the same in all reference frames

Implications of the Lorentz Transformations [R&L, p. 108 - 111]

We showed earlier that the Lorentz transformations and the inverse transformations are

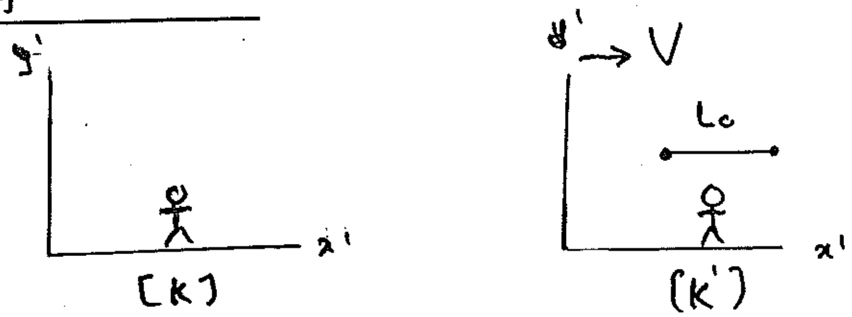
Lorentz

$$x' = \Gamma(x - Vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \Gamma\left(t - \frac{Vx}{c^2}\right)$$

Inverse - Lorentz

$$x = \Gamma(x' + Vt)$$
$$y = y'$$
$$z = z'$$
$$t = \Gamma\left(t' + \frac{Vx'}{c^2}\right)$$

1.) Lorentz Contraction

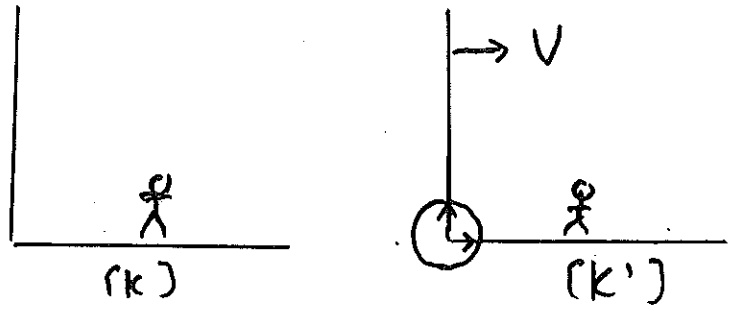


The length of an object like a measuring stick is measured by an observer in [K'] and he/she assign a length L_o to it. The same measurement carried out by a stationary observer in [K] reveals the following:

$$L_o = x_2' - x_1'$$
$$= \Gamma(x_2 - Vt) - \Gamma(x_1 - Vt)$$
$$= \Gamma(x_2 - x_1)$$
$$= \Gamma L$$
$$\Rightarrow L = \frac{1}{\Gamma} L_o \qquad \Gamma > 1 \qquad L < L_o$$

Lorentz contraction

2.) Time Dilation



Consider a stationary clock positioned at the origin of a moving reference system. Suppose the clock registers a time interval T_0. The corresponding time interval in the lab frame is

$$T = t_2 - t_1$$

$$= \Gamma\left(t_2' + \frac{V x'}{c^2}\right) - \Gamma\left(t_1' + \frac{V x'}{c^2}\right)$$

$$= \Gamma\,(t_2' - t_1') \qquad x' = 0 \text{ since clock is positioned at centre}$$

$$= \Gamma\, T_0$$

One can see that more time passed between the events in [K] than in [K']. Therefore one can say that time runs slower in a moving reference frame.

Aberration and Beaming

The Lorentz and inverse-Lorentz transformations are

Lorentz

$$x' = \Gamma(x - Vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \Gamma\left(t - \frac{Vx}{c^2}\right)$$

inverse-Lorentz

$$x = \Gamma(x' + Vt')$$
$$y = y'$$
$$z = z'$$
$$t = \Gamma\left(t' + \frac{Vx'}{c^2}\right)$$

Exercise: Show that by using these transformations the velocity of objects in the two frames are

$$v_x = \frac{v_x' + V}{1 + \frac{V v_x'}{c^2}} \qquad v_x' = \frac{v_x - V}{1 - \frac{V v_x}{c^2}}$$

$$v_y = \frac{v_y'}{\Gamma\left(1 + \frac{V v_x'}{c^2}\right)} \qquad v_y' = \frac{v_y}{\Gamma\left(1 - \frac{V v_x'}{c^2}\right)}$$

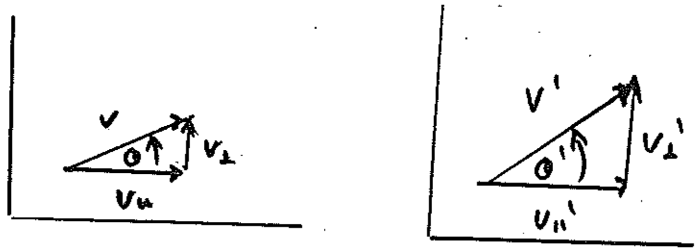
$$v_z = \frac{v_z'}{\Gamma\left(1 + \frac{V v_x'}{c^2}\right)} \qquad v_z' = \frac{v_z}{\Gamma\left(1 - \frac{V v_x'}{c^2}\right)}$$

or

$$v_\parallel = \frac{v_\parallel' + V}{\left(1 + \frac{V v_\parallel'}{c^2}\right)} \qquad v_\parallel' = \frac{v_\parallel - V}{\left(1 - \frac{V v_\parallel}{c^2}\right)}$$

$$v_\perp = \frac{v_\perp'}{\Gamma\left(1 + \frac{V v_x'}{c^2}\right)} \qquad v_\perp' = \frac{v_\perp}{\Gamma\left(1 - \frac{V v_\parallel}{c^2}\right)}$$

The aberration is then



(K) (K')

$$\tan\theta = \frac{v_\perp}{v_\parallel}$$

$$= \frac{\left[\dfrac{v_\perp'}{\gamma\left(1 + \frac{V v_\parallel'}{c^2}\right)}\right]}{\left[\dfrac{v_\parallel' + V}{1 + \frac{V v_\parallel'}{c^2}}\right]}$$

$$= \frac{v_\perp'}{\gamma\,(v_\parallel' + V)}$$

$$= \frac{v'\sin\theta'}{\gamma(v'\cos\theta' + V)}$$

For $v' = c$

$$\tan\theta = \frac{c\sin\theta'}{\gamma(c\cos\theta' + V)}$$

$$= \frac{c\sin\theta'}{\gamma c\left(\cos\theta' + \frac{V}{c}\right)}$$

$$= \frac{\sin\theta'}{\gamma\left(\cos\theta' + V/c\right)}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta'}{\gamma\left(\cos\theta' + V/c\right)}$$

Then from

$$v_{\parallel} = \frac{v_{\parallel}' + V}{\left(1 + \frac{V v_{\parallel}'}{c^2}\right)}$$

for $v_{\parallel}' \to c_{\parallel} = c\cos\theta'$ (velocity of light same in all reference frames)

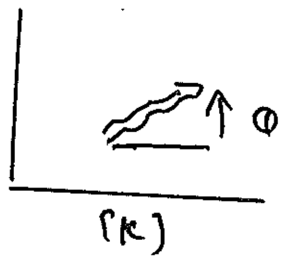
$$c\cos\theta = \frac{c\cos\theta' + V}{\left(1 + \frac{Vc\cos\theta'}{c^2}\right)}$$

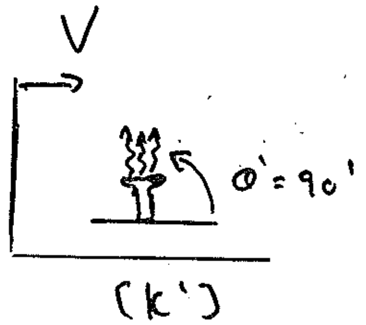
$$\cos\theta = \frac{\left(\cos\theta' + \frac{V}{c}\right)}{\left(1 + \frac{V\cos\theta'}{c}\right)}$$

Then from previous eqn

$$\sin\theta = \frac{\sin\theta'}{\Gamma\left(\cos\theta' + \frac{V}{c}\right)} \times \frac{\left(\cos\theta' + \frac{V}{c}\right)}{\left(1 + \frac{V\cos\theta'}{c}\right)}$$

$$= \frac{\sin\theta'}{\Gamma\left(1 + \frac{V}{c}\cos\theta'\right)}$$



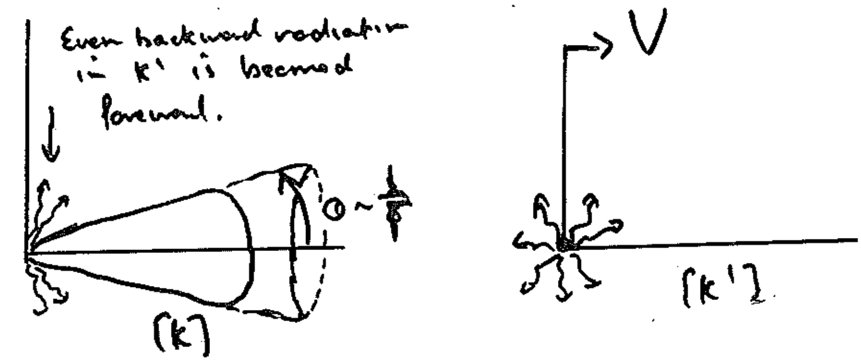


$$\sin\theta = \frac{1}{\Gamma} \qquad [\sin(90°) = 1 \quad \cos(90°) = 0]$$

From (K) the light gets beamed into the forward direction. So for $\Gamma >> 1$

$$\sin\theta \approx \theta \approx \frac{1}{\Gamma}$$

If an isotropic light source is placed on [K']
then we get the following beamed radiation
pattern



This is relativistic beaming. The effect is
severe if $V \to c$ so that $\Gamma \to \infty$.

<u>Note that:</u>

The angular limits of the forward cone of radiation is:

$$\sin\theta = \frac{1}{\Gamma}$$

$$\Rightarrow \quad (1-\cos^2\theta)^{1/2} = \frac{1}{\Gamma}$$

$\Rightarrow$ This is satisfied by condition

$$\beta^2 = \cos^2\theta$$

$$\beta = \cos\theta$$

So that if:

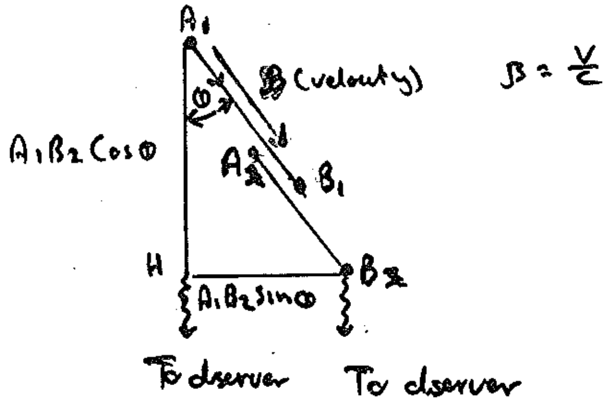
$$\cos\theta \to \beta \quad \Rightarrow \quad \theta \sim \frac{1}{\Gamma}$$

$$\text{For: } \beta \to 1 \quad \Rightarrow \quad \theta \to 0 \qquad (\Gamma \to \infty)$$

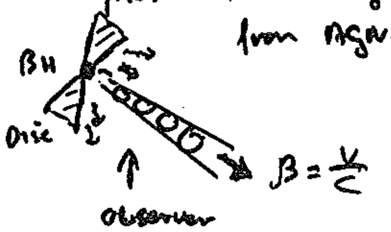
The Moving Bar

[K']

Let us consider a moving bar of proper dimension $l' = L_0$, moving in the direction of it's length at a velocity βc, and an angle θ with respect to the line of sight



Insert: This can represent a series of radio emitting plasma blobs in jet from AGN.



The length of the bar in the reference frame K (Lab frame) is

$$l = \frac{l'}{\Gamma} \quad \text{[Contracts in direction of motion]}$$

Let's assume that a photon emitted from A_1 reaches H in the time interval Δt_e. After Δt_e the front end of the bar reaches B_2. Then photons from the back end (emitted at A_1) and front end (emitted at B_2) can reach the observer simultaneously since the light paths are the same, i.e

Distance: (H to Observer) = Distance: (B_2 to Observer)

The length B_1B_2 is

$$B_1B_2 = c\beta\Delta t_e \qquad \beta = \frac{v}{c} \text{ (velocity of bar)}$$

The length A_1H is

$$A_1H = c\Delta t_e$$

Therefore

(1) $A_1H = A_1B_2 \cos\theta$

(2) $HB_2 = A_1B_2 \sin\theta$

Then

$$\Delta t_e = \frac{A_1H}{c}$$
$$= \frac{A_1B_2\cos\theta}{c}$$

We have to keep in mind that photons emitted from A_1 have a distance A_1H further to travel than photons emitted at B_2. Therefore from

$$\Delta t_e = \frac{A_1B_2\cos\theta}{c}$$

one can deduce

$$A_1B_1 = A_1B_2 - B_1B_2$$
$$= A_1B_2 - \beta c\,\Delta t_e$$
$$= A_1B_2 - \beta c\left[\frac{A_1B_2\cos\theta}{c}\right]$$
$$= A_1B_2\left[1 - \beta\cos\theta\right]$$

Hence

$$A_1B_2 = \frac{A_1B_1}{\left[1-\beta\cos\theta\right]}$$
$$= \frac{l}{\left[1-\beta\cos\theta\right]} \qquad l = \frac{l'}{\Gamma}$$
$$= \frac{l'}{\Gamma\left[1-\beta\cos\theta\right]} = \delta_D l'$$

<u>Doppler factor</u>: $\delta_D = \dfrac{1}{\Gamma(1-\beta\cos\theta)}$

Then one can write

$$\Delta te = \frac{A_1 B_2 \cos\theta}{c}$$

$$= \frac{l' \cos\theta}{\Gamma c(1-\beta\cos\theta)}$$

However in reality one would see a projection of $A_1 B_2$, i.e

$$H B_2 = A_1 B_2 \sin\theta$$

$$= \left[\frac{l'}{\Gamma(1-\beta\cos\theta}\right] \sin\theta$$

$$= \delta_D l' \sin\theta$$

The observed length depends on the viewing angle. The observed length reaches a maximum l' for

$$\cos\theta = \beta$$

$$\Rightarrow \quad H.B_2 = \frac{l'}{\Gamma(1-\beta\cos\theta)} \sin\theta \qquad \sin\theta = \left[1-\cos^2\theta\right]^{1/2}$$

$$= \left[\frac{l'}{\Gamma(1-\beta\cos\theta)}\right] (1-\cos^2\theta)^{1/2}$$

$$= \frac{l'}{\Gamma(1-\beta^2)} \underbrace{(1-\beta^2)^{1/2}}_{\frac{1}{\Gamma}}$$

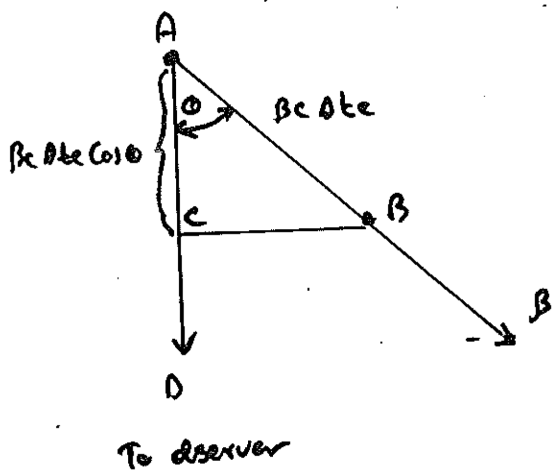
$$= \frac{l'}{\Gamma^2 \underbrace{(1-\beta^2)}_{\frac{1}{\Gamma^2}}}$$

$$= \frac{\Gamma^2}{\Gamma^2} l'$$

$$= l'$$

Time - Frequency

Consider a light source (Lamp) with a velocity $v = \beta c$ with respect to the line of sight



Insert: This can again be linked to a radio emitting blobs ejected by a SMBH in AGN. These blobs emit and then fade away.

In the frame of the lamp (K') it remains on for a time $\Delta t_e'$. The corresponding time interval in K [Lab frame] should be

$$\Delta t_e = \Gamma \Delta t_e'$$

However if we, the observers, use photons to measure this time interval we must consider that the first and last photons are emitted from different locations and that there light travel distance to observer is different. To measure Δt_e, i.e the time interval between arrival of the first and last photons we proceed as follows:

The first photon is emitted at A and the last at B. If these photons are measured in K, then the path of the light source AB is

$$AB = \beta c \Delta t_e$$
$$= \beta c \Gamma \Delta t_e'$$

While the lightsource moved from A to B the photon emitted when the lightsource was at A travelled a distance

$$AD = c\,\overbrace{\Delta t_e}^{\text{measured by observer in (k).}}$$

When the lightsource reaches B and emits the last photon, the first photons are at D. Along the line of sight, the first and last photons, i.e the ones emitted at A and B are separated by the distance CD. The corresponding time interval measured between first and last photon is:

$$\Delta t_a = \frac{CD}{c}$$

$$= \frac{AD - AC}{c}$$

$$= \frac{AD}{c} - \frac{AC}{c}$$

$$= \frac{c\Delta t_e}{c} - \frac{\beta c \Delta t_e \cos\theta}{c}$$

$$= \Delta t_e (1 - \beta\cos\theta)$$

$$= \Gamma \Delta t_e' (1 - \beta \cos\theta)$$

$$= \frac{\Delta t_e'}{\delta_D}$$

$$\delta_D = \frac{1}{\Gamma(1 - \beta\cos\theta)}$$

If θ is small and the velocity relativistic

$$(1 - \beta\cos\theta) \rightarrow (1 - \beta) \qquad \cos\theta \rightarrow 1$$

Then :

With $\cos\theta \to 1$ and $\Gamma = \frac{1}{(1-\beta^2)^{1/2}} = \frac{1}{[(1+\beta)(1-\beta)]^{1/2}}$

$$\Rightarrow \Gamma(1-\beta\cos\theta) = \frac{1-\beta}{[(1+\beta)(1-\beta)]^{1/2}}$$

$$= \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

Hence

$$\Gamma(1-\beta\cos\theta) \to \left(\frac{1-\beta}{1+\beta}\right)^{1/2} < 1$$

$$\frac{1}{\delta_D} = \Gamma(1-\beta\cos\theta) < 1 \qquad (\cos\theta \to 1)$$

$$\delta_D > 1$$

Therefore we measure a time contraction since

$$\Delta t_a = \frac{\Delta t_e'}{\delta_D}$$

$$\underbrace{\Delta t_a < \Delta t_e}_{\text{extreme Doppler boosting (Bunching)}} \qquad \delta_D > 1$$

Notice that we recover the usual time dilation

$$\Delta t_a = \Gamma \Delta t_e'$$

for $\theta = 90°$ because all photons have to travel the same distance towards observer.

Now since the frequency is the inverse of time

$$\left(\frac{1}{\Delta t_a}\right) = \left(\frac{1}{\Delta t_e'/\delta_D}\right)$$

$$\nu = \delta_D \nu' = \frac{\nu'}{\Gamma(1-\beta\cos\theta)}$$

The Doppler factor.

$$\delta_D = \frac{1}{\Gamma(1-\beta\cos\theta)}.$$

- $\frac{1}{\Gamma}$ the usual relativistic correction
- $\left(\frac{1}{1-\beta\cos\theta}\right)$ corresponds to the usual Doppler effect.

The Doppler factor δ_D is the result of the competition of the two terms above. For $\theta = \frac{\pi}{2}$ the Doppler term $(1-\beta\cos\theta) \Rightarrow 1$ and

$$\delta_D = \frac{1}{\Gamma}$$

For small θ the factor $\left(\frac{1}{1-\beta\cos\theta}\right) \Rightarrow \infty$ and dominates the $\frac{1}{\Gamma}$ factor. For $\cos\theta = \beta$ ($\sin\theta = \frac{1}{\Gamma}$) we have

$$\delta_D = \frac{1}{\Gamma(1-\beta\cos\theta)}.$$

$$= \frac{\Gamma^2}{\Gamma}$$

$$= \Gamma$$

For $\theta = 0°$ we have the following

$$\delta_D = \frac{1}{\Gamma(1-\beta\cos\theta)}$$

$$= \frac{1}{\Gamma(1-\beta)} = \frac{[(1-\beta)(1+\beta)]^{1/2}}{(1-\beta)} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2}$$

$$\Rightarrow \delta_D = \frac{[(1+\beta)(1+\beta)]^{1/2}}{[(1-\beta)(1+\beta)]^{1/2}} = \frac{1+\beta}{(1-\beta^2)^{1/2}} = \Gamma(1+\beta)$$

Insert:

$\theta = 180°$

$$\delta_D = \frac{1}{\Gamma(1-\beta\cos\theta)}$$

$$= \frac{1}{\Gamma(1+\beta)} = \frac{[(1+\beta)(1-\beta)]^{1/2}}{1+\beta} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

This can again be written as:

$$\delta_D = \left[\frac{(1-\beta)(1-\beta)}{(1+\beta)(1-\beta)}\right]^{1/2} = \frac{[(1-\beta)(1-\beta)]^{1/2}}{(1-\beta^2)^{1/2}} = \Gamma(1-\beta)$$

$\theta = 0°$: $\quad \delta_D = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} = \Gamma(1+\beta)$

$\theta = 180°$: $\quad \delta_D = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} = \Gamma(1-\beta)$

Hence:

$\underline{\theta = 0°}$: Source moves towards observer

$$\nu = \delta_D \nu'$$

$$= \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \nu' \qquad \beta = \frac{v}{c}$$

$\nu > \nu'$ (Blue-shift)

$\underline{\theta = 180°}$: Source moves away from observer

$$\nu = \delta_D \nu'$$

$$= \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \nu'$$

$\nu < \nu'$ (Red-shift)

Abberation & Transformation of Solid angle

Exercise: make use of the Lorentz and inverse - Lorentz transformations to show that

$$\sin\theta = \frac{\sin\theta'}{\Gamma(1+\beta\cos\theta')} \qquad \sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)}$$

$$\cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'} \qquad \cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}$$

Note that if $\theta' = 90°$ and $\beta \to 1$ $[\sin\theta' = 1 ; \cos\theta' = 0]$

$$\sin\theta = \frac{1}{\Gamma}$$

$$\theta = \frac{1}{\Gamma} \quad \text{for} \quad \boxed{\Gamma \gg 1 \Rightarrow \sin\theta \to \theta}$$

Therefore radiation is beamed in the forward direction, i.e in the direction of motion.

The definition of solid angle is

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \sin\theta \, d\theta$$

$$= 2\pi \left[\frac{\sin\theta'}{\Gamma(1+\beta\cos\theta')}\right]\left[\frac{d\theta'}{\Gamma(1+\beta\cos\theta')}\right]$$

$$= 2\pi \frac{\sin\theta' \, d\theta'}{[\Gamma(1+\beta\cos\theta')]^2}$$

$$= \frac{2\pi \sin\theta' \, d\theta'}{[\Gamma(1+\beta\cos\theta')]^2}$$

But

$$d\Omega' = 2\pi \sin\theta' \, d\theta'$$

$$d\Omega = d\Omega' \frac{1}{\left[\Gamma(1+\beta\cos\theta')\right]^2}$$

But we showed earlier that

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}$$

$$d\Omega = d\Omega' \left[\frac{1}{\Gamma\left(1+\beta\left[\frac{\cos\theta-\beta}{1-\beta\cos\theta}\right]\right)}\right]^2$$

$$= d\Omega' \left[\frac{1}{\Gamma\left(\frac{1-\beta\cos\theta+\beta\cos\theta-\beta^2}{1-\beta\cos\theta}\right)}\right]^2$$

$$= d\Omega' \left[\frac{1}{\Gamma\left(\frac{1-\beta^2}{1-\beta\cos\theta}\right)}\right]^2$$

$$= \frac{d\Omega'}{\Gamma^2} \frac{(1-\beta\cos\theta)^2}{(1-\beta^2)^2}$$

$$= \Gamma^2 (1-\beta\cos\theta)^2 \, d\Omega' \qquad \frac{1}{(1-\beta^2)^2} = \Gamma^4$$

$$= \frac{d\Omega'}{\delta_D^2}$$

Transformation of Intensity

The specific intensity is the energy per unit surface, time, frequency and solid angle

$$I_\nu = h\nu \times \frac{dN}{dA\,dt\,d\nu\,d\Omega}$$

To transform this to the rest frame k' we need to transform the various quantities dA, dt, $d\nu$ and $d\Omega$ to the rest frame. Since $dN = dN'$ (invariant).

We know from the Doppler equation that

$$\Delta t = (1-\beta\cos\theta)\,\Gamma\,\Delta t'$$

$$= \frac{\Delta t'}{\delta_D}$$

Also

$$\nu = \frac{\nu'}{\Gamma(1-\beta\cos\theta)}$$

$$= \delta_D\,\nu'$$

$$\Rightarrow d\nu = \delta_D\,d\nu'$$

The surface area stays invariant since it is $\perp$ to the direction of the light source

$$\Rightarrow dA = dA'$$

Therefore the intensity is

$$I_\nu = h\nu \frac{dN}{dA\, dt\, d\nu\, d\Omega}$$

$$= h(\delta_D \nu') \times \frac{dN'}{dA' \left(\frac{dt'}{\delta_D}\right)(\delta_D d\nu')\left(\frac{d\Omega'}{\delta_D^2}\right)}$$

$$= \delta_D^3 h\nu' \frac{dN'}{dA'\, dt'\, d\nu'\, d\Omega'}$$

$$= \delta_D^3\, I'_{\nu'}$$

$$I'_{\nu'} = h\nu' \frac{dN'}{dA' dt' d\nu' d\Omega'}$$

The total intensity or brightness

$$I = \int I_\nu\, d\nu$$

$$= \int \delta_D^3 I'_{\nu'}\, d\nu$$

$$= \int \delta_D^3 I'_{\nu'} \left(\frac{d\nu}{d\nu'}\right) d\nu'$$

We showed earlier

$$d\nu = \delta_D\, d\nu'$$

$$I = \delta_D^3 \int I'_{\nu'}\, \delta_D\, d\nu'$$

$$= \delta_D^4 \int I'_{\nu'}\, d\nu'$$

$$= \delta_D^4\, I' \qquad I' = \int I'_{\nu'}\, d\nu'$$

The fourth power of δ_D^4 comes from the transformation of frequency, time and twice for solid angle.

Insert: Transformation of observed flux

$$dF = \frac{dG}{dA\,dt}$$

$$dF_\nu = \frac{dF}{d\nu} = \frac{dG}{dA\,d\nu\,dt}$$

$$dF_\nu\,(\text{obs}) = \frac{dG}{dA\,dt\,d\nu} \qquad\qquad dA = D^2\,d\Omega$$

$$= \frac{dG}{D^2\,d\Omega\,dt\,d\nu}$$

But we showed that

- $d\nu = \delta_D\,d\nu' \Rightarrow d(h\nu) = \delta_D\,d(h\nu') \Rightarrow dE = \delta_D\,dE'$
- $dt = \frac{1}{\delta_D}\,dt'$
- $d\Omega = \frac{1}{\delta_D^2}\,d\Omega'$

Then

$$dF_\nu\,(\text{obs}) = \frac{dG}{D^2\,d\Omega\,dt\,d\nu}$$

$$= \frac{\delta_D\,dG'}{D^2\left(\frac{1}{\delta_D^2}d\Omega'\right)\left(\frac{1}{\delta_D}dt'\right)\left(\delta_D\,d\nu'\right)}$$

$$= \delta_D^3\,\frac{dG'}{D^2\,d\Omega'\,dt'\,d\nu'}$$

$$= \delta_D^3\,\frac{dG'}{4\pi D^2\,dt'\,d\nu'} \qquad\qquad d\Omega' = 4\pi$$

$$= \delta_D^3\,\frac{(dG'/dt')}{4\pi D^2\,d\nu'}$$

$$= \delta_D^3\left(\frac{L'}{d\nu'}\right)\times\frac{1}{4\pi D^2}$$

$$= \delta_D^3\,\frac{L'_{\nu'}}{4\pi D^2} = \delta_D^3\,F_{\nu'}'(\text{iso})$$

Emissivity

The frequency integrated emissivity j is the energy emitted per unit time, volume and solid angle. We generally have that

$$I = \int_{\Delta R} j \, dr$$

where ΔR is the length of the emitting region containing the emitting particles. The emissivity transforms like

$$j = h\nu \frac{dN}{dV \, dt \, d\Omega}$$

Notice that the spatial volume transforms like

$$dV = dA \, dL$$

$$= (\gamma_D dL')(dA')$$

$$= \gamma_D \, dA' dL'$$

$$= \gamma_D \, dV'$$

Then

$$j = h\nu \frac{dN}{dV dt d\Omega}$$

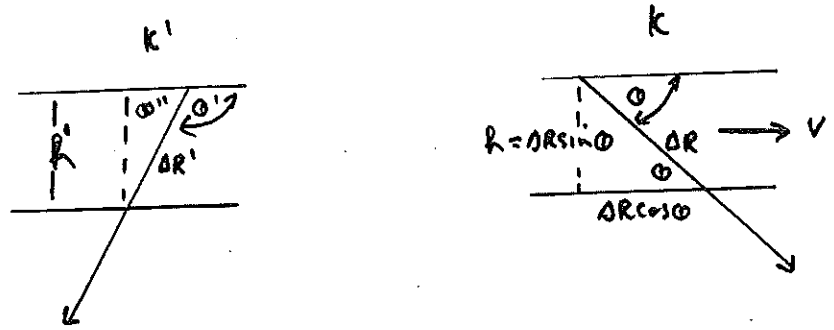
$$= h(\gamma_D \nu') \frac{dN'}{(\gamma_D dV')\left(\frac{dt'}{\gamma_D}\right)\left(\frac{d\Omega'}{\gamma_D^2}\right)}$$

$$= \gamma_D^3 h\nu' \frac{dN'}{dV' dt' d\Omega'}$$

$$= \gamma_D^3 j'$$

The emissivity transforms with one power less than intensity or brightness.

To understand the transformation properties of of emissivity consider a slab with plasma moving with a velocity parallel to the walls (e.g Fig 3.6 Chiselfini).



The observer in k will measure a ΔR which depends on his/her viewing angle. In k' the same path has a different length because of the abberation of light. The height of the slab is

$$h = h' \quad (\perp \text{ to motion, invariant})$$

In k the light ray travels a distance

$$h = \Delta R \sin\theta$$

$$\Delta R = \frac{h}{\sin\theta}$$

In k' we have

$$\Delta R' = \frac{h'}{\sin\theta'}$$

Let $\theta'' = 180 - \theta'$

$$\begin{aligned} h' &= \Delta R' \sin\theta'' \\ &= \Delta R' \sin(180 - \theta') \\ &= \Delta R' [\sin 180^\circ \cos\theta' - \cos 180^\circ \sin\theta'] \\ &= \Delta R' [-(-1)\sin\theta'] \\ &= \Delta R' \sin\theta' \end{aligned}$$

since $\sin 180 = 0$; $\cos 180^\circ = -1$

The abberation formula states

$$\sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)}$$

$$= \delta_D \sin\theta$$

we can write

$$\Delta R' = \frac{h'}{\sin\theta'}$$ [see previous transparency]

$$= \frac{h}{\delta_D \sin\theta}$$

$$= \frac{1}{\delta_D}\ \frac{h}{\sin\theta}$$

$$= \frac{1}{\delta_D}\ \Delta R$$ [see inserted notes for explanation]

Therefore the column of plasma contributing to the emission, for $\delta_D > 1$ [$\beta \to 1$ and $\theta \to 0°$] is less than what an observer in K would guess by measuring ΔR.

For simplicity, assume a homogeneous plasma

$$I = j\,\Delta R$$

$$= (\delta_D^3\, j') \, (\delta_D\, \Delta R')$$

$$= \delta_D^4\, j'\, \Delta R'$$

$$= \delta^4\, I'$$

$$I = j\,\Delta R = \delta_D^4\, j'\, \Delta R'$$

$$\dot{j} = \delta_D^4 \frac{\Delta R'}{\Delta R} \dot{j}'$$

$$= (\delta_D^4)(\frac{1}{\delta_D}) \dot{j}'$$

$$= \delta_D^3 \dot{j}'$$

The corresponding transformation for specific emissivity is

$$\dot{j}_\nu = \left(\frac{\dot{j}}{d\nu}\right) \qquad d\nu = \frac{d\nu}{d\nu'} d\nu'$$

$$= \frac{\delta_D^3 \dot{j}'}{\left(\frac{d\nu}{d\nu'}\right) d\nu'}$$

$$= \left(\frac{d\nu'}{d\nu}\right) \delta_D^3 \frac{\dot{j}'}{d\nu'} \qquad \frac{d\nu'}{d\nu} = \frac{1}{\delta_D}$$

$$= \left(\frac{1}{\delta_D}\right)(\delta_D^3) \frac{\dot{j}'}{d\nu'}$$

$$= \delta_D^2 \frac{\dot{j}'}{d\nu'}$$

$$= \delta_D^2 \dot{j}_{\nu'}$$

Insert:

We showed that two observers, one in [K'] and the other in [K] will measure different distances a lightray has to traverse through a moving slab of gas. This will impact on the absorbtion coefficient for both frames.

In [K]

$$\tau_\nu = \alpha_\nu \Delta R = \alpha_\nu \frac{h}{\sin\theta}$$

In [K']

$$\tau_\nu' = \alpha_\nu' \Delta R' = \alpha_\nu' \frac{h'}{\sin\theta'}$$

But: $\tau_\nu = \tau_\nu'$ [τ is just a number and therefore a Lorentz invariant]

$$\alpha_\nu' \Delta R' = \alpha_\nu \Delta R$$

$$\alpha_\nu' \frac{h'}{\sin\theta'} = \alpha_\nu \frac{h}{\sin\theta}$$

$$\frac{\alpha_\nu' h'}{\alpha_\nu h} = \frac{\sin\theta'}{\sin\theta}$$

$h' = h$ [$\perp$ to motion]
[Eq. 3.16 p. 44]

$$\left(\sin\theta' = \frac{\sin\theta}{\Gamma(1 - \frac{V}{c}\cos\theta)} = \delta_D \sin\theta\right)$$

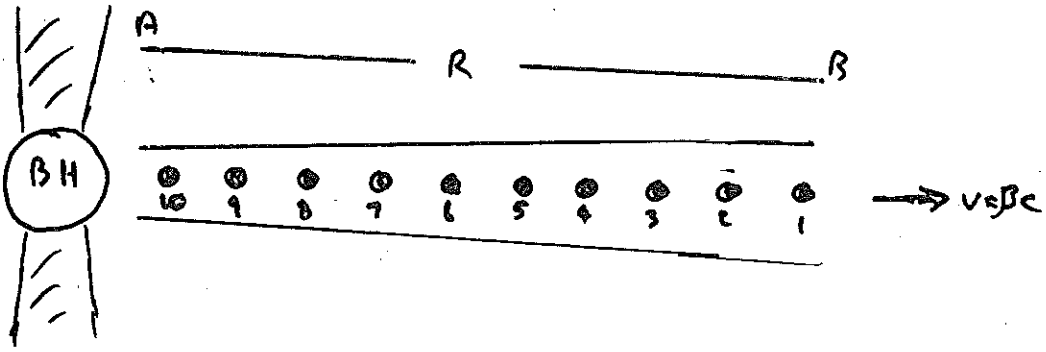
$$\frac{\alpha_\nu'}{\alpha_\nu} = \frac{\sin\theta'}{\sin\theta} = \delta_D$$

$$\alpha_\nu' = \delta_D \alpha_\nu$$

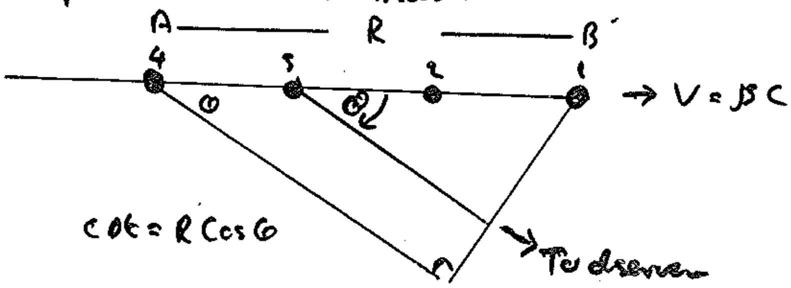
or

$\alpha_\nu = \frac{1}{\delta_D}\alpha_\nu'$ [If a observer [illegible] a larger ΔR [K] [illegible] is small

Example: Blobby Jet from AGN — Sikora et al. 1997 (ApJ 484, 108)



Let say that a distance R from the apex of a jet (in k) there are at any given time N emitting blobs (10 in this example). These blobs move with velocity $v=\beta c$ along jet. For the sake of clarity we assume that beyond R these emitting blobs switch off. If the viewing angle $\theta = 90°$ the photons emitted by the blobs have to travel the same distance to observer. Hence the observer will see all 10 blobs. But if $\theta < 90°$, the photons produced by the rear blobs must travel a larger distance in order to reach the observer, therefore they have to be emitted before the photons emitted by the front blobs. Let's explain using 4 blobs:



Since the blobs only emit for a finite time, they switch-off after moving a distance R the observer will see less blobs. Consider a viewing angle $\theta < 90°$. Photons emitted by blob 4 to reach the same light travel distance to observer, were born (emitted) when blob 4 was at A and blob 1 at B

These photons travels a distance $R\cos\theta$ in a time Δt. During the same time blob 4 travels a distance

$$\Delta R = \beta c \Delta t$$

in the forward direction. The fraction (f) of the blobs that can be seen is

$$f = \frac{R - \Delta R}{R}$$

$$= 1 - \frac{\Delta R}{R}$$

$$= 1 - \frac{\beta c \Delta t}{R} \qquad c\Delta t = R\cos\theta$$

$$= 1 - \frac{\beta R \cos\theta}{R}$$

$$= 1 - \beta\cos\theta$$

This is the usual Doppler factor. Then we can write

$$f = \frac{\Gamma(1-\beta\cos\theta)}{\Gamma}$$

$$= \frac{1}{\Gamma}\,\frac{1}{\delta_D}$$

The moral of the story: Even if the flux of a single blob is boosted by δ_D^4 and the jet has N-blobs the $\underbrace{\text{total flux}}_{\hookrightarrow N\text{-blobs}}$ is not boosted by $\delta_D^4 \times$ flux/blob.

This is because the observer sees less blobs for $\theta < 90°$!!
This shows that the 'viewing angle plays a very important role in jet emission.

Brightness Temperature:

The brightness temperature is a quantity used especially in radio astronomy. It is defined as the temperature an emitting source in the R-J limit should have to produce an R-J intensity I

$$I_\nu = \frac{2\nu^2}{c^2} k T_B \qquad (h\nu \ll kT)$$

Remember: We showed earlier

$$T_B = T(1 - e^{-\tau_\nu})$$

$$T_B \longrightarrow T \qquad \tau_\nu \gg 1$$

$$T_B \longrightarrow T\tau_\nu \qquad \tau_\nu \ll 1$$

Hence

$$T_B = \frac{I_\nu}{2k} \frac{c^2}{\nu^2}$$

Since

$$F(\nu) = I_\nu \Delta\Omega$$

$$= I_\nu \pi \theta_s^2$$

$$I_\nu = \frac{F(\nu)}{\pi \theta_s^2}$$

$$\Rightarrow \quad T_B = \frac{\left(\frac{F(\nu)}{\pi \theta_s^2}\right)}{2k} \frac{c^2}{\nu^2}$$

Insert:

Since

$$T_b \propto \frac{I_\nu}{\nu^2} \qquad I_\nu = \frac{F_\nu}{\Delta\Omega}$$

$$\propto \frac{F_\nu}{\nu^2 \Delta\Omega}$$

$$\propto \frac{F_\nu}{\nu^2 (\Delta\theta_{obs})^2}$$

$$T_b \propto \frac{1}{(\Delta\theta_{obs})^2}$$

If the real emitting region inside the source is smaller than $\Delta\theta_{obs}$, this will impact on T_b. For

$$\Delta\theta_{emit} \approx \alpha\, \Delta\theta_{obs}. \qquad (\alpha < 1)$$

$$T_b(\theta_{emit}) \propto \frac{1}{(\Delta\theta_{emit})^2}$$

$$\propto \left(\frac{1}{\alpha^2}\right) T_b(\theta_{obs})$$

$$T_b(\theta_{emit}) > T_b(\theta_{obs})$$

There are two ways to measure θ_s

① <u>VLBI Observations</u>

With VLBI measurements one can resolve even the smallest sources. One can show that with the VLBI the resolving power is

$$\theta \approx \frac{\lambda}{D}$$

- $\theta \approx 0.007'' \left(\frac{\lambda}{21\,cm}\right)\left(\frac{D}{6000\,km}\right)^{-1}$ $\{21\,cm = 1.4\,GHz\}$
- $\theta \approx 10^{-4}{}'' \left(\frac{\lambda}{0.3\,cm}\right)\left(\frac{D}{6000\,km}\right)^{-1}$ $\boxed{0.3\,cm = 100\,GHz}$

In this case most sources will be resolved, i.e. one can observe different parts of the source. In this case the relation between the brightness temperature measured in K and K' respectively is

$$\left.\begin{aligned} I_\nu &= \delta_D^3\, I_{\nu'}' \\ \nu &= \delta_D\, \nu' \end{aligned}\right\} \text{shown earlier}$$

$$\begin{aligned} T_B &= \frac{I_\nu}{2k}\,\frac{c^2}{\nu^2} \\ &= \frac{\delta_D^3\, I_{\nu'}'}{2k}\,\frac{c^2}{(\delta_D\nu')^2} \\ &= \delta_D\,\frac{I_{\nu'}'}{2k}\,\frac{c^2}{(\nu')^2} \\ &= \delta_D\, T_B' \end{aligned}$$

$$T_B' = \frac{I_{\nu'}'}{2k}\,\frac{c^2}{(\nu')^2}$$

This can also be expressed in terms of the observed flux from the source

$$F(\nu) = \frac{L}{4\pi D^2}$$

$$= \frac{4\pi \int j_\nu \, dV}{4\pi D^2} \quad \text{[isotropic emission]}$$

$$= \frac{\int j_\nu \, dV}{D^2}$$

$$= \frac{1}{D^2} \int \delta_D^2 j'_{\nu'} \, \delta_D \, dV'$$

$$= \frac{\delta_D^3}{D^2} \int j'_{\nu'} \, dV'$$

$$= \delta_D^3 \frac{\int j'_{\nu'} \, dV'}{D^2}$$

$$= \delta_D^3 F'(\nu')$$

Then the brightness temperature is

$$T_B = \frac{F(\nu)}{2\pi k \Theta_s^2} \frac{c^2}{\nu^2}$$

$$= \frac{\delta_D^3 F'(\nu')}{2\pi k \Theta_s^2} \frac{c^2}{(\delta_D \nu')^2}$$

$$= \delta_D \frac{F'(\nu')}{2\pi k \Theta_s^2} \frac{c^2}{(\nu')^2}$$

$$= \delta_D T_B'$$

where $T_B' = \frac{F'(\nu')}{2\pi k \Theta_s^2} \frac{c^2}{(\nu')^2}$

If the brightness temp of an extragalactic radio source is abnormally high ($T \sim 10^{11}$ K) then significant beaming must be present - we can put constraint on that.

(2) Variability

If the source is variable then we can constrain the size by demanding that the variability timescale be shorter than the light crossing time R/c, where R is the typical size of the emitting region. The angular size

$$\Theta_s = \frac{(c\,\Delta t_{var})^2}{d_A^2}$$ d_A = angular distance

$$\Delta t_{var} = \frac{\Delta t'_{var}}{\delta_D}$$

Therefore

$$T_B = \frac{F(\nu)}{2\pi k\,\Theta_s^2}\,\frac{c^2}{\nu^2}$$

$$= \frac{\delta_D^3 F'(\nu')}{2\pi k}\,\frac{1}{\left(\frac{c\Delta t_{var}}{d_A}\right)^2}\,\frac{c^2}{(\delta_D \nu')^2}$$

$$= \frac{\delta_D^3 F'(\nu')}{2\pi k}\,\frac{d_A^2}{\left(c\,\frac{\Delta t'_{var}}{\delta_D}\right)^2}\,\frac{c^2}{(\delta_D \nu')^2}$$

$$= \delta_D^3\,\frac{F'(\nu')}{2\pi k}\,\frac{1}{\left(\frac{c\Delta t'_{var}}{d_A}\right)^2}\,\frac{c^2}{(\nu')^2}$$

$$> \delta_D^3\, T_B'$$

Here we assume that

$$\left(\frac{c\,\Delta t'_{var}}{d_A}\right) < \Theta_s$$

$$T_B > \delta_D^3\, T_B'$$

The same apply to a variable source. For abnormally high T_b, we can put constraint on δ_D.

Flux from moving sources

We showed that

$$I_\nu = \delta_D^3 \, I'_{\nu'}$$

Consider a BB source with intrinsic brightness $I'_{\nu'}$ in its own rest frame. The brightness can be explained in terms of a BB spectrum

$$I'_{\nu'} = \frac{2h\nu'^3}{c^2}\left(\frac{1}{e^{\frac{h\nu'}{kT'}} - 1}\right)$$

A stationary observer in K' sees a BB spectrum with characteristic temperature T'. The relation between T' (in K') and T_B (in K) is

$$T_B = \delta \, T_B'$$

But the corresponding brightness in the lab frame is

$$I_\nu = \delta_D^3 \, I'_{\nu'}$$

This gives

$$I_\nu = \delta_D^3 \, \frac{2h\nu'^3}{c^2}\left(\frac{1}{e^{h\nu'/kT'} - 1}\right)$$

$$= \delta_D^3 \, \frac{2h\left(\frac{\nu}{\delta_D}\right)^3}{c^2}\left[e^{\frac{h\left(\frac{\nu}{\delta_D}\right)}{k\left(\frac{T}{\delta_D}\right)}} - 1\right]^{-1}$$

$$= \frac{2h\nu^3}{c^2}\left(\frac{1}{e^{\frac{h\nu}{kT}} - 1}\right)$$

A BB remains a BB in all reference frames.

Suppose the source in K' emits a power law

$$I'_{\nu'} \propto \nu'^{-\alpha} \qquad \text{[Synchrotron or IC]}$$

$$\begin{aligned} I_\nu &= \delta_D^3 \, I'_{\nu'} \\ &\propto \delta_D^3 \, \nu'^{-\alpha} \\ &\propto \delta_D^3 \left(\frac{\nu}{\delta_D}\right)^{-\alpha} \\ &\propto \delta_D^{3+\alpha} \, \nu^{-\alpha} \end{aligned}$$

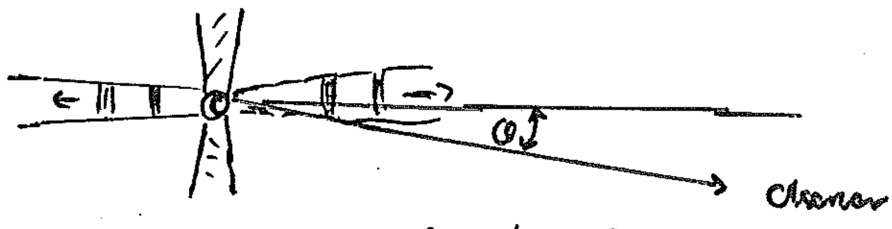
with

$$\delta_D = \frac{1}{\Gamma(1-\beta\cos\theta)}$$

Therefore

$$I_\nu \propto \left[\frac{1}{\Gamma(1-\beta\cos\theta)}\right]^{3+\alpha} \nu^{-\alpha}$$

We can use this relation to compare the relative brightness of two components of a jet from a AGN ejected in opposite directions



Towards observer : $I_{1\nu}(\theta=0) \propto \left[\frac{1}{\Gamma(1-\beta)}\right]^{3+\alpha} \nu^{-\alpha}$
$\theta \to 0$

Away from observer : $I_{2\nu}(\theta=180°) \propto \left[\frac{1}{\Gamma(1+\beta)}\right]^{3+\alpha} \nu^{-\alpha}$
$\theta \to 180°$

The ratio

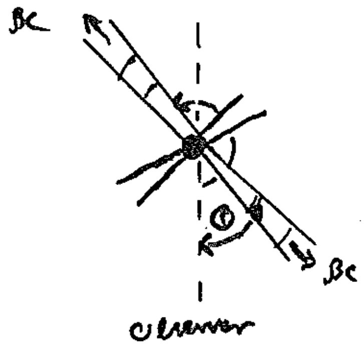
$$\frac{I_{1\nu}(\Theta=0^\circ)}{I_{2\nu}(\Theta=180^\circ)} \propto \left(\frac{1+\beta}{1-\beta}\right)^{3+\alpha}$$

$$\frac{I_{1\nu}(\Theta=0^\circ)}{I_{2\nu}(\Theta=180^\circ)} \gg 1 \qquad \beta \to 1$$

$$I_{1\nu}(\Theta=0^\circ) \gg I_{2\nu}(\Theta=180^\circ) \qquad \beta \to 1$$

The emission from the part of the jet moving towards observer ($\Theta \to 0^\circ$) is highly Doppler boosted and therefore much brighter than the part of the jet moving away from the observer.

Let's use a more general example. When the approaching and receding velocity is not directly towards ($\Theta = 0^\circ$) and away ($\Theta = 180^\circ$)



The intensity for the part of the jet towards observer

$$I_{1\nu} \propto \left[\frac{1}{\Gamma(1-\beta\cos\Theta)}\right]^{3+\alpha} \nu^{-\alpha}$$

For a source receding the angle will be $\Theta_r = (\pi + \Theta)$

$$\cos(\pi+\Theta) = \cos\pi\cos\Theta - \sin\pi\sin\Theta$$

$$\cos(\pi+\Theta) = -\cos\Theta \qquad [\cos\pi = -1\,;\ \sin\pi = 0]$$

$$I_{2\nu} \propto \left[\frac{1}{\Gamma(1-\beta\cos\Theta_r)}\right]^{3+\alpha} \nu^{-\alpha} \qquad \cos\Theta_r = \cos(\pi+\Theta) = -\cos\Theta$$

This gives

$$I_{2\nu} \propto \left[\frac{1}{\Gamma(1+\beta\cos\theta)}\right]^{3+\alpha} \nu^{-\alpha}$$

Therefore the ratio

$$\frac{I_{1\nu}\text{ (towards)}}{I_{2\nu}\text{ (away)}} \propto \frac{\left[\frac{1}{\Gamma(1-\beta\cos\theta)}\right]^{3+\alpha} \nu^{-\alpha}}{\left[\frac{1}{\Gamma(1+\beta\cos\theta)}\right]^{3+\alpha} \nu^{-\alpha}}$$

$$\propto \left[\frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right]^{3+\alpha}$$

Therefore

$$\frac{I_{1\nu}}{I_{2\nu}} \gg 1 \quad \text{for} \quad \beta\cos\theta \to 1$$

The limits (angular) for the forward cone of beamed radiation is:

$$\sin\theta = \frac{1}{\Gamma}$$

$$\sin^2\theta = \frac{1}{\Gamma^2}$$

$$1-\cos^2\theta = \frac{1}{\Gamma^2}$$

$$\cos^2\theta = 1 - \frac{1}{\Gamma^2}$$

$$= 1-(1-\beta^2)$$

$$= \beta^2$$

$$\cos\theta = \beta$$

In this limit:

$$\frac{I_{1\nu}\,(\text{towards})}{I_{2\nu}\,(\text{away})} \propto \left[\frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right]^{3+\alpha}$$

$$\propto \left(\frac{1+\beta^2}{1-\beta^2}\right)^{3+\alpha} \qquad \cos\theta \to \beta$$

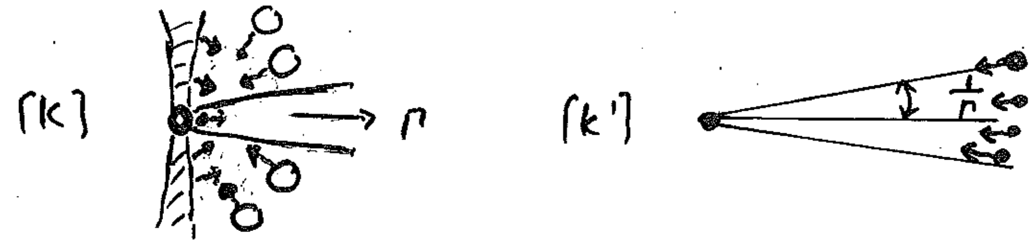
$$\propto \left(\Gamma^2(1+\beta^2)\right)^{3+\alpha} \qquad 1-\beta^2 = \frac{1}{\Gamma^2}$$

$$\propto \left(2\Gamma^2\right)^{3+\alpha} \qquad \beta \to 1$$

The approaching component is much brighter than the receding component. This is why most of the jets from AGN appear to be one-sided.

Motion in a Homogeneous Radiation Field

Jets in AGN's plunge through an external radiation field produced by the accretion disc as well as the line emitting clouds orbiting the BH.



For an external observer in [K] a relativistic blob ejected by the BH will be exposed to an isotropic radiation field from the disc and line emitting regions. However for the blob the abberation eqn's tells us that the radiation will come mainly from the forward direction, i.e from within a semi-cone with angle

$$\theta' = \frac{1}{\Gamma}$$

Let's quantify this :

$$\sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)}$$

For $\theta \to 90°$ in K

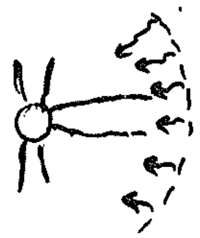
$$\sin\theta' = \frac{1}{\Gamma} \qquad [\sin\theta = 1 \quad \cos\theta = 0]$$

So even the photons coming in from the sides of the jet (i.e from disc for example) will appear as if they come from the front, i.e from within

$$\theta' \to \frac{1}{\Gamma}$$

This means that the energy density of the radiation, as is seen by the relativistic blob in its own frame will be boosted by a factor $\sim \Gamma^2$. We can show this.

As an example let's take the jet (Fig 3.8 Ghisellini) surrounded by the disc and BLR's. The jet has a bulk Lorentz factor Γ and associated velocity $V_{jet} \sim \beta c$. For simplicity assume that the BLR photons are produced by the surface of a sphere with radius R and that the jet is within that sphere.



Let's also assume the radiation is monochromatic at some frequency ν_0 (in K). The co-moving observer in K' will see photons coming in from a cone with semi-aperture $\theta' \sim \frac{1}{\Gamma}$.

Using the Doppler-relation, inverting it now just to take into account that the observer is now in K'

$$\nu'_{obs} = \frac{\nu_{em}}{\Gamma(1-\beta\cos\theta')}$$

Photons coming from the front: $\theta' \to 0$

$$= \frac{\nu_{em}}{\Gamma(1-\beta)}$$

Remark:

We showed earlier that

$$\nu_{obs}[K] = \frac{\nu_{em}[K']}{\Gamma(1-\beta\cos\underbrace{\theta}_{[K]})}$$

$$= \frac{(1-\beta^2)^{1/2}}{1-\beta}\nu_{em}$$

$$\Gamma = \frac{1}{(1-\beta^2)^{1/2}}$$

Now the observer is in $[K']$, hence:

$$= \frac{[(1-\beta)(1+\beta)]^{1/2}}{(1-\beta)}\nu_{em}$$

$$\nu_{obs}[K'] = \frac{\nu_{em}[K]}{\Gamma(1-\beta\cos\underbrace{\theta'}_{[K']})} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2}\nu_{em}$$

$$= \frac{1+\beta}{[(1-\beta)(1+\beta)]^{1/2}}\nu_{em} = \frac{1+\beta}{(1-\beta^2)^{1/2}}\nu_{em} = \Gamma(1+\beta)\nu_{em}$$

Photons that the observer in k see as coming in from the side, i.e $\theta \to 90°$ will be seen in k' as coming from the angle

$$\sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)}$$

Photons coming from the side

$$= \frac{1}{\Gamma} \qquad (\sin 90° = 1\ ;\ \cos 90° = 0)$$

$$\theta' \sim \frac{1}{\Gamma} \quad \text{for} \quad \Gamma \gg 1$$

The resultant frequency a co-moving observer in k' will measure is

$$\nu'_{obs} = \frac{\nu_{em}}{\Gamma(1-\beta\cos\theta')}$$

$\sin\theta' = \frac{1}{\Gamma} \qquad \theta = 90°$

$\sin^2\theta' = \frac{1}{\Gamma^2}$

$1-\cos^2\theta' = \frac{1}{\Gamma^2}$

$1-\cos^2\theta' = 1-\beta^2$

$\cos\theta' = \beta$

$$= \frac{\nu_{em}}{\Gamma(1-\beta^2)}$$

$$\approx \frac{\Gamma^2}{\Gamma}\nu_{em}$$

$$= \Gamma\nu_{em}$$

So even the photons coming in from the disc from large angles $\theta \to \pi/2$ are Doppler boosted for the co-moving observer.

From the reference frame k' (for a co-moving observer) the Doppler factor will be

$$\delta_D' = \frac{1}{\Gamma(1-\beta\cos\theta')}$$

A comoving observer in K' will then see the intensity or brightness of the BLR photons to be

$$I' = \delta_D^{4'} I$$

We know that the brightness and energy density are related

$$u_\nu(\Omega) = \frac{I_\nu}{c}$$

$$u_\nu = \int u_\nu(\Omega)\, d\Omega$$

$$u = \int u_\nu\, d\nu$$

$$= \int \int u_\nu(\Omega)\, d\Omega\, d\nu$$

$$= \int \left[\int \frac{1}{c} I_\nu\, d\nu \right] d\Omega$$

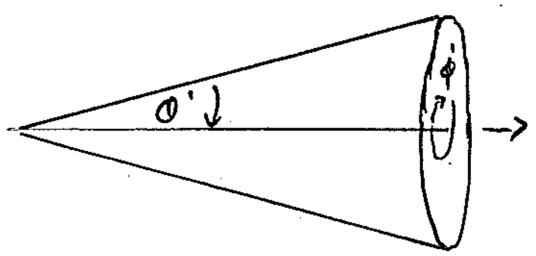
$$= \frac{1}{c} \int I\, d\Omega$$

Now let's apply this to determine the energy density a co-moving observer will measure in K'. The solid angle in K' is

$$d\Omega' = \sin\theta'\, d\theta'\, d\phi'$$

$$= d\phi\, \sin\theta'\, d\theta'$$

↑ ϕ is invariant since it is in plane $\perp$ to motion of jet.



Therefore we can write

$$\frac{d}{d\theta'}(\cos\theta') = -\sin\theta'$$

$$d\cos\theta' = -\sin\theta'\, d\theta'$$

$$\sin\theta'\, d\theta' = -d\cos\theta'$$

$$\theta' \Rightarrow \left(0, \frac{1}{\Gamma}\right)$$

$$\cos\theta' \Rightarrow (1, \beta)$$

Therefore

$$u' = \frac{1}{c}\int I'\, d\Omega'$$

$$= \frac{1}{c}\int_{\Omega'} I'\,(d\phi'(-d\cos\theta'))$$

$$= -\frac{1}{c}\int_{\phi} d\phi \int_{1}^{\beta} I'\, d\cos\theta' \qquad \phi' = \phi$$

$$= \frac{2\pi}{c}\int_{\beta}^{1} I'\, d\cos\theta'$$

$$= \frac{2\pi}{c}\int_{\beta}^{1} (\gamma_D')^4\, I\, d\cos\theta'$$

$$= \frac{2\pi}{c}\int_{\beta}^{1} \frac{I}{[\Gamma(1-\beta\cos\theta')]^4}\, d\cos\theta'$$

$$= \frac{2\pi I}{\Gamma^4 c}\int_{\beta}^{1} \frac{d\cos\theta'}{[1-\beta\cos\theta']^4}$$

The integral can be solved as follows:

$$\frac{d}{d\cos\theta'}\left[1-\beta\cos\theta'\right]^{-3} = -3\left[1-\beta\cos\theta'\right]^{-4}(-\beta)$$

$$= \frac{3\beta}{[1-\beta\cos\theta']^4}$$

This is

$$u' = \frac{2\pi I}{\Gamma^4 c} \int_\beta^1 \frac{d\cos\theta'}{(1-\beta\cos\theta')^4}$$

$$= \frac{2\pi I}{\Gamma^4 c} \int_\beta^1 \frac{1}{3\beta}\, d(1-\beta\cos\theta')^{-3}$$

$$= \frac{2\pi I}{\Gamma^4 c} \left[\frac{1}{3\beta}\left[1-\beta\cos\theta'\right]^{-3}\right]_\beta^1$$

$$= \frac{2\pi I}{\Gamma^4 c} \left[\frac{(1-\beta\cos\theta')^{-3}}{3\beta}\right]_\beta^1$$

$$= \frac{2\pi I}{\Gamma^4 c} \left[\frac{(1-\beta)^{-3}}{3\beta} - \frac{(1-\beta^2)^{-3}}{3\beta}\right]$$

$$= \frac{2\pi I}{\Gamma^4 c} \left[\frac{\overbrace{\left[(1-\beta)(1+\beta)\right]^{-3}}^{\Gamma^6}}{3\beta(1+\beta)^{-3}} - \frac{\overbrace{(1-\beta^2)^{-3}}^{\Gamma^6}}{3\beta}\right]$$

$$= \frac{2\pi I}{\Gamma^4 c}\, \Gamma^6 \left[\frac{(1+\beta)^3}{3\beta} - \frac{1}{3\beta}\right]$$

$$= \frac{2\pi \Gamma^2 I}{c} \left[\frac{(1+3\beta+3\beta^2+\beta^3)-1}{3\beta}\right]$$

$$= \frac{2\pi I}{c}\, \Gamma^2 \left[\frac{3\beta+3\beta^2+\beta^3}{3\beta}\right]$$

$$= \frac{2\pi I}{c}\, \Gamma^2 \left(1+\beta+\frac{\beta^2}{3}\right)$$

$$= \Gamma^2 u \left(1+\beta+\frac{\beta^2}{3}\right)$$

Notice that

$$u = \frac{2\pi I}{c} = \frac{1}{2} \times \frac{4\pi I}{c}$$ [one side of disc.]

For

$$u' = \Gamma^2 u \left(1 + \beta + \frac{\beta^2}{3}\right)$$

for $\beta \to 1$ [relativistic jet]

$$u' = \Gamma^2 u \left[1 + 1 + \frac{(1)^2}{3}\right]$$

$$= \frac{7}{3} \Gamma^2 u$$

So the co-moving observer will see the total energy density of the background radiation field boosted by a factor

$$\alpha = \frac{7}{3} \Gamma^2$$

from the energy density in the Lab frame.

Exercise: Show that if the same analysis is performed over a whole sphere $\theta' = [0, \pi] \Rightarrow \cos\theta' = [1, -1]$ we get

$$u' = \frac{4}{3} \Gamma^2 u_{iso} \qquad u_{iso} = \frac{4\pi I}{c}$$

Hint: for isotropic emission

$$\theta' = 0 \qquad \cos\theta' = 1$$

$$\theta' = 180^\circ \qquad \cos\theta' = -1$$

Hence

$$u' = \frac{2\pi I}{\Gamma^4 c} \int_{-1}^{+1} \frac{d\cos\theta'}{(1 - \beta\cos\theta')^4}$$

$$= \frac{2\pi I}{\Gamma^4 c} \left[\frac{(1 - \beta\cos\theta')^{-3}}{3\beta}\right]_{-1}^{+1} = \frac{4}{3}\Gamma^2 \times \frac{4\pi I}{c} = \frac{4}{3}\Gamma^2 u_{iso}$$

Insert:

Determine the energy density in the direction opposite to the direction of motion.

Now $\theta_1' = \frac{\pi}{2}$, which means $\cos\theta_1' = 0$. When $\theta_2' = \pi$ then $\cos\theta_2' = -1$. Therefore

$$U' = \frac{2\pi I}{\Gamma^4 c} \int_{-1}^{0} \frac{d\cos\theta'}{(1-\beta\cos\theta')^4}$$

$$= \frac{2\pi I}{\Gamma^4 c} \left[\frac{(1-\beta\cos\theta')^{-3}}{3\beta} \right]_{-1}^{0}$$

$$= \frac{1}{2\Gamma^4} \left(\frac{4\pi I}{c}\right) \left[\frac{(1-\beta\cos\theta')^{-3}}{3\beta} \right]_{-1}^{0}$$

$$= \frac{1}{2\Gamma^4} U_{iso} \left[\frac{1}{3\beta} - \frac{(1+\beta)^{-3}}{3\beta} \right]$$

$$= \frac{1}{6\beta\Gamma^4} U_{iso} \left[1 - (1+\beta)^{-3} \right]$$

$$= \frac{1}{6\beta\Gamma^4} U_{iso} \left[1 - \frac{1}{(1+\beta)^3} \right]$$

$$= \frac{1}{6\beta\Gamma^4} U_{iso} \left[\frac{(1+\beta)^3 - 1}{(1+\beta)^3} \right]$$

$$= \frac{1}{6\beta\Gamma^4} U_{iso} \left[\frac{1+3\beta+3\beta^2+\beta^3-1}{(1+\beta)^3} \right]$$

$$= \frac{1}{6\beta\Gamma^4} U_{iso} \left[\frac{3\beta+3\beta^2+\beta^3}{(1+\beta)^3} \right]$$

$$= \frac{1}{6\beta\Gamma^4} U_{iso} \frac{3\beta\left(1+\beta+\frac{\beta^2}{3}\right)}{(1+\beta)^3}$$

$$= \frac{1}{2\Gamma^4} U_{iso} \frac{\left(1+\beta+\frac{\beta^2}{3}\right)}{(1+\beta)^3}$$

One can see that for:

$\beta \to 1$ $\quad \Gamma >> 1$ (very large)

$$U' \approx \frac{1}{2\Gamma^4} \, U_{iso} \, \frac{(1+1+\frac{1}{3})}{(2)^3}$$

$$= \frac{1}{2\Gamma^4} U_{iso} \frac{(7/3)}{(8)}$$

$$= \left(\frac{7}{2\times 3\times 8}\right) \frac{1}{\Gamma^4} \, U_{iso}$$

$$= \frac{7}{48} \, \frac{1}{\Gamma^4} \, U_{iso}$$

It's clear that $U' << U_{iso}$ (void in energy density)

Even if $\beta << 1$ and $\Gamma \sim 1$

$$\Rightarrow \boxed{U' \approx \frac{1}{2} U_{iso}}$$

For non-relativistic ($\Gamma \approx 1$) motion and $\left(0 < \theta' < \frac{1}{\Gamma}\right)$

$$\Rightarrow \boxed{\begin{aligned} U' &\approx \frac{7}{3} \, \Gamma^2 \, U_{iso} \\ &\approx \frac{7}{3} \, U_{iso} \end{aligned}}$$

For BB spectrum

$$\boxed{\begin{aligned} U &= a\,T^4 \\ U' &= a(T')^4 \end{aligned}}$$

Behind:

$U' \approx \frac{1}{2} U_{iso}$

$\Rightarrow a(T')^4 \approx \frac{1}{2} \, a \, T_{iso}^4$

$\Rightarrow (T')^4 = \frac{1}{2} \, T_{iso}^4$

$\Rightarrow (T') = \left(\frac{1}{2}\right)^{1/4} T_{iso}$ $\quad (T' \leq T_{iso})$

In front:

$U' \approx \frac{7}{3} \, U_{iso}$

$\Rightarrow a(T')^4 \approx \frac{7}{3} \, a \, T_{iso}^4$

$\Rightarrow (T')^4 \approx \frac{7}{3} \, T_{iso}^4$

$\Rightarrow T' \approx \left(\frac{7}{3}\right)^{1/4} T_{iso}$ $\quad (T' \geq T_{iso})$

Remark:

For an observer moving through isotropic photon field the temp of CMB is cooler behind than infront.

Monochromatic Flux

A monochromatic flux in K is seen in K' at different frequencies between $\Gamma \nu_0$ and $(1+\beta)\Gamma \nu_0$ with slope

$$F'(\nu') \propto \nu'^2$$

This is shown in the following way:

We showed earlier that photons from directions

1.) $[\theta = 90° \text{ in } K] \Rightarrow [\cos\theta' = \beta \text{ in } K'] \Rightarrow \nu'_{obs} = \Gamma \nu_{em}$
2.) Photons with $\theta' = 0°$ in K' $\Rightarrow \nu'_{obs} = \Gamma(1+\beta)\nu_{em}$

Therefore the energies of photons within the lightcone

$$0 \le \theta' \le \frac{1}{\Gamma}$$

$$\underbrace{\Gamma \nu_{em}}_{\theta' = \frac{1}{\Gamma}} \le \nu'_{obs} \le \underbrace{\Gamma(1+\beta)\nu_{em}}_{\theta' = 0}$$

We already showed that

$$I'_{\nu'} = \delta_0'^3 I_\nu$$

$$= \left(\frac{\nu'}{\nu}\right)^3 I_\nu \qquad \left[\delta' = \frac{\overset{\text{observed}}{\nu'}}{\underset{\text{emitted}}{\nu}}\right]$$

The flux at a specific frequency is

$$F'(\nu') = \int I'_{\nu'}\, d\Omega'$$

$$= \int I'_{\nu'} \sin\theta'\, d\theta'\, d\phi' \qquad d\phi' = d\phi$$

Like before, changing integration variables

$$\sin\theta'\, d\theta' = -d\cos\theta'$$
$$= -d\mu' \qquad\qquad \mu' = \cos\theta'$$

Then

$$F'(\nu') = 2\pi \int_1^\beta \left(\frac{\nu'}{\nu}\right)^3 I_\nu \,(-d\cos\theta')$$
$$= 2\pi \int_\beta^1 \left(\frac{\nu'}{\nu}\right)^3 I_\nu \, d\cos\theta'$$
$$= 2\pi \int_{\mu_1'}^{\mu_2'} \left(\frac{\nu'}{\nu}\right)^3 I_\nu \, d\mu'$$

The integral is over those μ' contributing to ν' in k'.
Now since

$$\frac{\nu'}{\nu} = \delta_D' = \frac{1}{\Gamma(1-\beta\cos\theta')}$$
$$= \frac{1}{\Gamma(1-\beta\mu')}$$

$$\Gamma(1-\beta\mu') = \frac{1}{\delta_D'}$$
$$\Gamma - \Gamma\beta\mu' = \frac{1}{\delta_D'}$$
$$-\Gamma\beta\mu' = \frac{1}{\delta_D'} - \Gamma$$
$$\mu' = \frac{-\Gamma}{-\Gamma\beta} - \frac{1}{\delta_D'\Gamma\beta}$$
$$= \frac{1}{\beta} - \frac{1}{\beta}\left(\frac{1}{\delta_D'\Gamma}\right)$$
$$= \frac{1}{\beta}\left[1 - \frac{1}{\delta_D'\Gamma}\right]$$

Then

$$\mu' = \frac{1}{\beta}\left(1 - \frac{1}{\delta_D' n}\right)$$
$$= \frac{1}{\beta}\left(1 - \frac{\nu}{\nu' n}\right) \qquad \delta_D' = \frac{\nu'}{\nu}$$

Therefore

$$d\mu' = \frac{1}{\beta}\left(-\frac{1}{n}\left(d\left(\frac{\nu}{\nu'}\right)\right)\right)$$
$$= -\frac{1}{\beta n}\, d\left(\frac{\nu}{\nu'}\right)$$
$$= -\frac{1}{\beta n}\,\frac{d\nu}{\nu'}$$

Since the intensity is monochromatic in k

$$I_\nu = I_0\, \delta(\nu - \nu_0)$$

Then

$$F'(\nu') = 2\pi \int_{\mu_1'}^{\mu_2'} \left(\frac{\nu'}{\nu}\right)^3 I_\nu \, d\mu'$$
$$= 2\pi \int_{\nu_1}^{\nu_2} \left(\frac{\nu'}{\nu}\right)^3 I_0\, \delta(\nu - \nu_0) \left[-\frac{d\nu}{\beta n \nu'}\right]$$
$$= \frac{2\pi I_0}{\beta n} \int_{\nu_2}^{\nu_1} \left(\frac{\nu'}{\nu}\right)^3 \left(\frac{1}{\nu'}\right) \delta(\nu - \nu_0)\, d\nu$$
$$= \frac{2\pi I_0}{n\beta}\,\frac{\nu'^2}{\nu_0^3} \qquad \left[n\nu_0 \le \nu' \le n(1+\beta)\,\nu_0\right]$$

The total integrated flux

$$F' = \int F'(\nu')\, d\nu'$$
$$= \frac{2\pi I_0}{n\beta}\,\frac{1}{\nu_0^3}\int \nu'^2\, d\nu'$$

This becomes

$$F' = \frac{2\pi I_0}{\Gamma\beta} \frac{1}{\nu_0^3} \left[\frac{\nu'^3}{3}\right]_{\Gamma\nu_0}^{\Gamma(1+\beta)\nu_0} \qquad [\Gamma\nu_0 \leq \nu' \leq \Gamma(1+\beta)\nu_0]$$

$$= \frac{2\pi I_0}{\Gamma\beta} \frac{1}{3\nu_0^3} \left[(\Gamma(1+\beta)\nu_0)^3 - (\Gamma\nu_0)^3\right]$$

$$= \frac{2\pi I_0}{\Gamma\beta} \frac{1}{3\nu_0^3} (\Gamma\nu_0)^3 \left[(1+\beta)^3 - 1\right]$$

$$= \frac{2\pi I_0}{3\beta} \Gamma^2 \left(1 + 3\beta + 3\beta^2 + \beta^3 - 1\right)$$

$$= \frac{2\pi I_0}{3\beta} \Gamma^2 \left(3\beta + 3\beta^2 + \beta^3\right)$$

$$= \frac{2\pi I_0}{3\beta} \Gamma^2 \, 3\beta\left(1 + \beta + \frac{\beta^2}{3}\right)$$

$$= 2\pi I_0 \Gamma^2 \left(1 + \beta + \frac{\beta^2}{3}\right)$$

$$= \Gamma^2\left(1 + \beta + \frac{\beta^2}{3}\right) F$$

Notice that

$$F = \int I \, d\Omega$$

$$= I \, 4\pi \qquad [\text{isotropic}]$$

$$\Rightarrow \quad F = 2\pi I_0 = \frac{1}{2} F_{iso}$$

This is because we only work in one hemisphere.

Superluminal Motion

In 1971 the VLBI began linking different radio telescopes over distances over several thousands of kilometres. The resolving power of a telescope is of the order

$$\alpha = \frac{\lambda}{D}$$

where λ is the wavelength and D the diameter of the telescope. This can also be for an interferometer the furthest distance between telescope units.

Observing at 1 cm with two telescopes seperated by 1000 km ($\sim 10^8$ cm) means that we can resolve structures

$$\alpha \approx \frac{1\,\mathrm{cm}}{10^8\,\mathrm{cm}}$$

$$\sim 10^{-8} \text{ radians}$$

$$\sim 10^{-8} \times (57.3 \text{ degrees} \times 3600 \text{ arc seconds})$$

$$\sim 2\times10^{-3} \text{ arc second}$$

$$\sim \text{milli - arc second}$$

The first VLBI observations revealed that the inner-jet of radio loud quasars are not homogeneous but blobby. Repeated observations showed that these blobs were not stationary but moved. Comparing radio maps that were taken at different times showed an angular displacement $\Delta\Theta$ between positions of these blobs.

By knowing the distance to these source through red shift of the spectral lines one can transform $\Delta\Theta$ to a distance

$$\Delta R = d\,\Delta\Theta$$

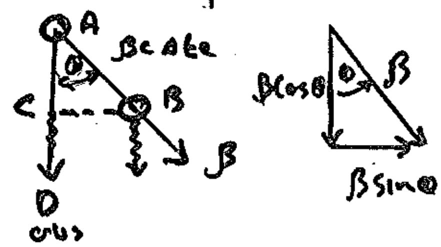
Dividing by the time interval Δt_a between the two radio maps one can obtain a velocity

$$V_{app} = \frac{d(\Delta\Theta)}{dt_A}$$

With some surprise in several objects this turned out to be larger than the speed of light c. Therefore these sources were called superluminal. However this is a misconception !!

What the observer measures is the projected displacement of a blob in the sky. The observed displacement is (see fig 8.4)

$$CB = \beta c\,\Delta t_e\,\sin\Theta$$



Dividing by $\Delta t_a = \Delta t_e\,(1-\beta\cos\Theta)$ we get an apparent velocity

$$V_{app} = \frac{\beta c\,\Delta t_e\,\sin\Theta}{\Delta t_e\,(1-\beta\cos\Theta)}$$

$$= \frac{\beta c\,\sin\Theta}{1-\beta\cos\Theta}$$

Projected velocity across the sky.

$$\beta_{app} = \frac{V_{app}}{c} = \frac{\beta\sin\Theta}{1-\beta\cos\Theta}$$

Consequence

For angle θ close to the line of sight this effect can be dramatic. The viewing angle for which this apparent velocity is a maximum can be obtained as follows:

$$\frac{\partial \beta_{app}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\beta \sin\theta}{1-\beta\cos\theta}\right) = 0$$

Exercise : Show that

$$\frac{\partial \beta_{app}}{\partial \theta} = \frac{\beta\cos\theta - \beta^2}{(1-\beta\cos\theta)^2}$$

Then

$$\frac{\partial \beta_{app}}{\partial \theta} = \frac{\beta\cos\theta - \beta^2}{(1-\beta\cos\theta)^2} = 0$$

This is satisfied if

$$\beta\cos\theta - \beta^2 = 0$$

$$\Rightarrow \cos\theta = \beta$$

Then for $\cos\theta = \beta$

$$\sin\theta = (1-\cos^2\theta)^{1/2} = (1-\beta^2)^{1/2} = \frac{1}{\Gamma}$$

$$\beta_{app} = \frac{\beta\sin\theta}{(1-\beta\cos\theta)} = \frac{\beta(1-\beta^2)^{1/2}}{(1-\beta^2)} = \frac{\beta}{(1-\beta^2)^{1/2}} = \Gamma\beta$$

Then for $\beta \to 1$ we know $\Gamma >> 1$ and

$$\beta_{app} = \Gamma \beta$$

$$\frac{V_{app}}{c} = \Gamma$$

$$V_{app} = \Gamma c$$

$$V_{app} > c \quad \text{for } \Gamma > 1$$

This is known as superluminal motion.

For 3C 273

$$V_{app} = 10\, c$$

Therefore

$$\Gamma = 10 \quad \text{(Bulk Lorentz factor)}$$

for 3C 273