Chapter 2

Particle acceleration

2.1 Propeller outflow and particle acceleration in AE Aquarii

In the previous chapter the enigmatic transient nature of the thermal and non-thermal emission from AE Aquarii was attributed to the propeller ejection of the mass transfer stream from the secondary star. The propeller ejection of material is the result of intense magnetohydrodynamic (mhd) interaction between the fast rotating white dwarf magnetosphere and the gas stream from the secondary. A detailed study (Meintjes & de Jager 2000) of the interaction between the magnetospheric propeller with slower orbiting gas near the circularization radius $R_c \sim 2 \times 10^{10}$ $(P_{orb}/9.88\ h)^{2/3}$ cm (Frank, King & Raine 1992) showed that a drag force is exerted on the white dwarf as a result of the magnetosphere's attempt to bring the gas into corotation. Shearing of the field will occur, resulting in the creation of a toroidal (azimuthal) magnetic field in the propeller zone (Meintjes & de Jager 2000) with strength of $B_\phi \sim (8\dot{m}v_{ff}/r^2)^{1/2}$ (v_{ff} is the free fall velocity), which is

$$B_\phi \approx 1000 \left(\frac{\dot{M}_{eject}}{5 \times 10^{17}\,\mathrm{g.s^{-1}}}\right)^{1/2} \left(\frac{M_1}{0.9M_\odot}\right)^{1/4} \left(\frac{r}{R_c}\right)^{-5/4} \mathrm{G}. \tag{2.1}$$

This results in the dissipation of power in the magnetosphere at a rate of $P_{mhd} \sim (B_\phi^2/8\pi)v_A R_c^2$ which is of the order of

$$P_{mhd} \approx 10^{34} \left(\frac{B_\phi}{1000\,\mathrm{G}}\right)^3 \left(\frac{P_{orb}}{9.88\,\mathrm{h}}\right)^{4/3} \left(\frac{n_g}{10^{10}\,\mathrm{cm^{-3}}}\right)^{-1/2} \mathrm{erg\ s^{-1}}, \tag{2.2}$$

similar to the spin down power as well as the rate of mass outflow from the system and where n_g is the density of the line emitting gas. The violent interaction between the magnetosphere and gas flow as a result of the dissipation of mhd power in this region may result in particle acceleration through various mechanisms, which will be discussed shortly.

Since the temperature in the magnetosphere of the white dwarf is nowhere below $T_m = 10^4$ K, the gas is highly conducting, with the Coulomb conductivity (Priest & Forbes 2000) nowhere below

$$\sigma \geq 3.22 \times 10^{12} \left(\frac{T_B}{10^4\ \mathrm{K}}\right)^{3/2} \mathrm{s}^{-1}. \tag{2.3}$$

The Coulomb conductivity of the fluid severely influences the nature of particle acceleration through electrodynamic processes and needs to be discussed. In the next section the properties of highly conducting fluids and the influence it has on particle acceleration in general is discussed.

Since the nature of the non-thermal emission is highly variable on relatively short time scales, the particles also need to be energized over similar time scales. Therefore, impulsive electrodynamic processes must definitely play an important role in the initial energizing of charged particles. If these particles are trapped in the magnetosphere, shock acceleration and magnetic pumping mechanisms can also be fundamental processes in continuously energizing this population. The best known regions of particle acceleration in astrophysics are the Sun and the geomagnetic tail. It is expected that processes at work here are also applicable to AE Aquarii. A qualitative discussion of these processes will now be presented (Parker 1976 for a review).

2.2 Acceleration in a highly conducting fluid

Consider electromagnetic fields in a highly conducting fluid with small velocity $\mathbf{v}$ ($v \ll c$). These fields can be deduced from Maxwell's equations. In the frame of the fluid, Ohm's law states that the current density is related to the conductivity of the fluid (σ) through J' = σ E' with $\sigma = 3.22 \times 10^6\ T_e^{3/2}$ (s^{-1}) representing the Coulomb conductivity. The transformation of the fields (Jackson 1975) from the laboratory frame or rest frame (i.e. K), to a reference frame comoving with the fluid (i.e. K') is given by

$$\mathbf{E}' = \gamma[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})]$$

and

$$\mathbf{B}' = \gamma[\mathbf{B} - \frac{1}{c}(\mathbf{v} \times \mathbf{E})].$$

where the primed quantities represent the fields in the comoving reference frame and the unprimed quantities represent the field in the laboratory system.

For $v/c \ll 1$, $\gamma \to 1$ and

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})$$

and

$$\mathbf{B}' = \mathbf{B} - \frac{1}{c}(\mathbf{v} \times \mathbf{E}).$$

If the conductivity is high, any current in the fluid's frame can be supported by $\mathbf{E}' \to 0$, resulting in

$$\mathbf{E} = -\frac{1}{c}(\mathbf{v} \times \mathbf{B}) \tag{2.4}$$

and

$$\begin{aligned}
\mathbf{B}' &= \mathbf{B} - \frac{1}{c}(\mathbf{v} \times \mathbf{E}) \\
&= \mathbf{B} - \frac{1}{c}(\mathbf{v} \times \left(-\frac{1}{c}(\mathbf{v} \times \mathbf{B})\right) \\
&= \mathbf{B} - \frac{1}{c^2}[\mathbf{v}(\mathbf{v}.\mathbf{B}) - \mathbf{B}(\mathbf{v}.\mathbf{v})] \quad ; \quad [\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}.\mathbf{c}) - \mathbf{c}(\mathbf{a}.\mathbf{b})] \\
&= \mathbf{B} \quad ; \quad [\text{neglect } \mathrm{O}(\mathrm{v/c})^{\mathbf{2}}\text{terms}] \qquad (2.5)
\end{aligned}$$

Let the characteristic scale of the field be L. Then if the characteristic time for adjustment in the field is $\tau = \mathrm{L/v}$, Ampère's law can be adapted as follows

$$\begin{aligned}
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\
c\nabla \times \mathbf{B} &\approx 4\pi\mathbf{J} + \frac{\mathbf{E}}{\tau} \\
&= 4\pi\mathbf{J} + \frac{\mathrm{v}}{\mathrm{L}}\mathbf{E} \qquad (2.6)
\end{aligned}$$

Now consider the dimensions of this equation

$$\left(\frac{\mathrm{cB}}{\mathrm{L}}\right) \approx 4\pi\mathrm{J} + \frac{v}{\mathrm{L}}E = 4\pi\mathrm{J} + \frac{v}{\mathrm{L}}\frac{\mathrm{vB}}{\mathrm{c}} = 4\pi\mathrm{J} + \frac{v^2}{\mathrm{c}^2}\left(\frac{\mathrm{cB}}{\mathrm{L}}\right)$$

The second term on the right $\left(\frac{v^2}{c^2}\left(\frac{cB}{L}\right)\right)$ can be ignored when compared to the left side. The result is that in a highly conducting fluid the $\frac{d\mathbf{E}}{dt}$ term can be left out of Ampère's law, illustrating that electric induction will not occur in a highly conducting fluid.

Therefore

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}. \tag{2.7}$$

Any current produced through the rotation of $\mathbf{B}$ in a conducting fluid or plasma is then of order

$$\mathrm{J} \approx \frac{\mathrm{cB}}{4\pi\mathrm{L}}$$

Now we get back to Ohm's law $\mathbf{J}' = \sigma\mathbf{E}'$ and

$$\mathbf{J}' = \mathbf{J} = \sigma\mathbf{E}' \quad [\text{where } \mathbf{J}' = \mathbf{J} \text{ in a slow moving plasma}]$$

Now

$$\begin{aligned}
\mathbf{J} &= \sigma\mathbf{E}' \\
&= \sigma[\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{B})] \quad ; \ \gamma \approx 1 \\
\Rightarrow \ \sigma\mathbf{E} &= \mathbf{J} - \sigma\frac{\mathbf{v}}{c}\times\mathbf{B}
\end{aligned}$$

so then

$$\begin{aligned}
\mathbf{E} &= \frac{\mathbf{J}}{\sigma} - \frac{\mathbf{v}}{c}\times\mathbf{B} \\
&= \frac{1}{\sigma}\left[\frac{c}{4\pi}(\nabla\times\mathbf{B})\right] - \frac{\mathbf{v}}{c}\times\mathbf{B} \ ; \ [\mathbf{J} \text{ from Eq.2.7}] \\
&= \frac{c}{4\pi\sigma}\nabla\times\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{B} \qquad (2.8)
\end{aligned}$$

This can now be substituted into Faraday's induction equation $\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$ to give

$$\begin{aligned}
\frac{\partial\mathbf{B}}{\partial t} &= -c\nabla\times\mathbf{E} \\
&= -c\nabla\times[\frac{c}{4\pi\sigma}\nabla\times\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{B}] \\
\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{v}\times\mathbf{B} - \nabla\times\eta\nabla\times\mathbf{B} \ ; \ [\eta = \frac{c^2}{4\pi\sigma}] \qquad (2.9)
\end{aligned}$$

where η represents a resistive coefficient for diffusion of the magnetic field relative to the fluid supporting it. The second term of Eq.2.9 can be called the diffusion term.

Consider the equations

$$\begin{aligned}
\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times\mathbf{v}\times\mathbf{B} - \nabla\times\eta\nabla\times\mathbf{B} \qquad (2.10) \\
\mathbf{E} &= \frac{c}{4\pi\sigma}\nabla\times\mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{B} \quad ; \ [\mathbf{E}\perp\mathbf{B}] \qquad (2.11)
\end{aligned}$$

Let us define the magnetic Reynolds number

$$R_m = \frac{Lv}{\eta}$$

Then

$$\begin{aligned}
\frac{\partial\mathbf{B}}{\partial t} &= \nabla\times(\mathbf{v}\times\mathbf{B} - \eta\nabla\times\mathbf{B}) \\
\frac{\partial\mathbf{B}}{\partial t} &\approx \frac{1}{L}(vB - \eta\frac{B}{L}) \\
&\approx \frac{vB}{L} - \frac{1}{R_m}\frac{vB}{L} \qquad (2.12)
\end{aligned}$$

Now for a highly conducting medium, $\sigma\to\infty$, hence $\eta\propto 1/\sigma\to 0$ and this results in $R_m\to\infty$. This means that the diffusion term is smaller in magnitude than the first term in Eq.2.9 by the reciprocal of this large Reynolds number.

Now we look at the electric field, perpendicular or parallel to the magnetic field $\mathbf{B}$

$$\begin{aligned}
\mathbf{E}_\perp &= -\frac{\mathbf{v}}{\mathrm{c}} \times \mathbf{B} + \frac{\mathrm{c}}{4\pi\sigma}\nabla \times \mathbf{B} \\
&= -\frac{\mathbf{v}}{\mathrm{c}} \times \mathbf{B} + \frac{\eta}{\mathrm{c}}\nabla \times \mathbf{B} \\
&= -\frac{\mathbf{v}}{\mathrm{c}} \times \mathbf{B} + \frac{1}{R_m}\frac{\mathrm{Lv}}{\mathrm{c}}\nabla \times \mathbf{B} \qquad (2.13)
\end{aligned}$$

If $R_m \to \infty$

$$\mathbf{E}_\perp \approx -\frac{1}{\mathrm{c}}\mathbf{v} \times \mathbf{B}$$

or $E_\perp \approx \frac{\mathrm{vB}}{\mathrm{c}}$.
The component parallel to the magnetic field gives

$$\mathbf{J}_{\|} = \sigma \mathbf{E}_{\|}$$

and thus

$$\begin{aligned}
\mathbf{E}_{\|} &= \mathbf{J}/\sigma \\
&= [\frac{\mathrm{c}}{4\pi}\nabla \times \mathbf{B}]_{\|}/\sigma \\
&= \frac{1}{R_m}\frac{\mathrm{Lv}}{\mathrm{c}}(\nabla \times \mathbf{B})
\end{aligned}$$

so then

$$E_{\|} \approx \frac{1}{R_m}\frac{\mathrm{Lv}}{\mathrm{c}}\frac{\mathrm{B}}{\mathrm{L}} \approx \frac{1}{R_m}\frac{\mathrm{vB}}{\mathrm{c}} \quad ; \quad [E_\perp \approx \frac{\mathrm{vB}}{\mathrm{c}}] \qquad (2.14)$$

We therefore see that $E_{\|} \approx \frac{1}{R_m}E_\perp$ and thus

$$E_{\|} \ll E_\perp$$

Therefore in a highly conducting fluid the parallel component of the electric field can be neglected. We proceed by looking at the motion of a free particle due to these calculated fields in a fluid with velocity $\mathbf{v}$. The equation of motion for such a particle of mass m_p, charge q and velocity $\mathbf{w}_p$ is

$$\begin{aligned}
\mathrm{m}_p\frac{d\mathbf{w}_p}{dt} &= \mathbf{F}_{\mathrm{Lorentz}} \\
&= \mathrm{q}[\mathbf{E} + \left(\frac{\mathbf{w}_p}{\mathrm{c}} \times \mathbf{B}\right)] \\
&\approx \mathrm{q}(\mathbf{E}_\perp + \frac{\mathbf{w}_p}{\mathrm{c}} \times \mathbf{B}) \\
&\approx \mathrm{q}\left(-\frac{1}{\mathrm{c}}\mathbf{v} \times \mathbf{B} + \frac{\mathbf{w}_p}{\mathrm{c}} \times \mathbf{B}\right) \\
\mathrm{m}_p\frac{d\mathbf{w}_p}{dt} &\approx \frac{\mathrm{q}}{\mathrm{c}}((\mathbf{w}_p - \mathbf{v}) \times \mathbf{B}) \qquad (2.15)
\end{aligned}$$

Then

$$m_p \frac{d\mathbf{w}_p}{dt}.\mathbf{w}_p = \frac{q}{c}((\mathbf{w}_p - \mathbf{v}) \times \mathbf{B}).\mathbf{w}_p$$

but we also have

$$\frac{d}{dt}(1/2 m_p w_p^2) = m_p \frac{d\mathbf{w}_p}{dt}.\mathbf{w}_p \tag{2.16}$$

Then we find

$$\begin{aligned} \frac{d}{dt}(1/2 m_p w_p^2) &= \frac{q}{c}((\mathbf{w}_p - \mathbf{v}) \times \mathbf{B}).\mathbf{w}_p \\ &= \frac{q}{c}[(\mathbf{w}_p \times \mathbf{B}).\mathbf{w}_p) - (\mathbf{v} \times \mathbf{B}).\mathbf{w}_p] \\ &= \frac{q}{c}[-(\mathbf{v} \times \mathbf{B}).\mathbf{w}_p] \quad ; \text{[since} (\mathbf{w_p} \times \mathbf{B}) \perp \mathbf{w_p}]. \end{aligned} \tag{2.17}$$

We can use the identity $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \mathbf{b}.(\mathbf{c} \times \mathbf{a}) = \mathbf{c}.(\mathbf{a} \times \mathbf{b})$.

and thus

$$\begin{aligned} \frac{d}{dt}(1/2 m_p w_p^2) &= -\frac{q}{c}(\mathbf{v} \times \mathbf{B}).\mathbf{w}_p \\ &= \mathbf{v}.[\frac{q}{c}(\mathbf{w}_p \times \mathbf{B})] \\ &= \mathbf{v}.\mathbf{F}_{p,\,\mathrm{Lorentz}} = vF_p \cos\theta \end{aligned} \tag{2.18}$$

The Lorentz force is always perpendicular to the particle velocity and the field direction. For q = -e and q = +e the maximum rate of energy transfer is $\frac{d}{dt}(1/2 m_p w_p^2)_{max} = vF_{p,Lor}$ with $\theta = \pi$ and 0 respectively. This means that the rate of kinetic energy increase is the same for the two charges, but the effective change in velocity is greater for the less massive electron. Now if θ is the angle between the fluid velocity and the B-field, the maximum acceleration of an electron occurs when the fluid tries to move the electron head-on against F_{Lor}. The B-field is frozen into the fluid and thus it can be said that the comoving fluid and field scatters the electron. This is basically a Fermi-type acceleration. In the case of the positive charge it is bumped by the field in the direction of the Lorentz force. We therefore see that any scattering of an electron relative to the fluid or magnetic field is likely to win the electron some energy or accelerate it.

2.3 Reduced conductivity and particle acceleration

The theoretical electrical conductivity for small current densities in an ionized gas is $\sigma = 10^7\, T^{3/2}$ esu . It has been suggested (Parker 1976) that this value may be greatly reduced at critical points in the system where current densities are high. The implied enhanced resistivity would then increase the dissipation *and the electric field* across the reduced conductivity region to keep the current constant (Meintjes & de Jager 2000) if it flows parallel to the magnetic field. If such regions of enhanced resistivity can be created, acceleration of particles can occur.

As an example, let the current density parallel to the B-field be so high that the electron drift velocity u exceeds the thermal ion velocity $v_{th} \approx \left(\frac{3kT}{m_i}\right)^{1/2}$. (The total translational (x, y, z) kinetic energy per particle is 3/2kT.) Then various micro-instabilities are invoked that greatly enhances the resistivity of the plasma and thus

$$E_{\parallel} = J_{\parallel}/\sigma \rightarrow \infty \quad (\sigma \rightarrow 0)$$

The current density is given by J = -$N_e$eu with N_e the electron density. Now if the magnetic field changes over a length scale l, from Ampère's law we get

$$J = \frac{cB}{4\pi l}$$

and so

$$\frac{cB}{4\pi l} = -N_e e u \Rightarrow u = -\frac{cB}{4\pi e N_e l} \tag{2.19}$$

If $N_e \rightarrow 0$, u can be larger than v_{th}.

So now let $u^2 > \frac{3kT}{m_i}$ or also

$$\frac{c^2B^2}{16\pi^2 e^2 N_e^2 l^2} > \frac{3kT}{m_i}.$$

The plasma frequency is $\omega_p = \left(\frac{4\pi N_e e^2}{m_e}\right)^{1/2}$.

Therefore

$$\begin{aligned}
\frac{c^2B^2}{4\pi^2 l^2 m_e}\left(\frac{m_e}{4\pi e^2 N_e}\right) &> \frac{3kT}{m_i} \\
\frac{c^2B^2}{4\pi^2 m_e l^2}\left(\frac{1}{\omega_p^2}\right) &> \frac{3kT}{m_i} \\
l^2\omega_p^2 &< \frac{c^2B^2}{4\pi^2 m_e}\frac{m_i}{3kT} \\
&< \left(\frac{\frac{B^2}{8\pi}}{3/2kT}\right)\frac{m_i}{m_e}c^2 \\
l\omega_p &< \left(\frac{P_{mag}}{P_{gas}}\right)^{1/2}\left(\frac{m_i}{m_e}\right)^{1/2} c \\
l\omega_p &< \left(\frac{m_i}{m_e}\right)^{1/2} c
\end{aligned} \tag{2.20}$$

The next step is to marginally quantify this condition which points to possibly enhanced resistivity. Currents produced by magnetic reconnection (discussed later) is dependent on the approach speed v of the diffusing field lines over the length l. Therefore if $v = \frac{\eta}{l}$, where η is the magnetic diffusion coefficient (see §2.1), then

$$l = \frac{\eta}{v} = \frac{\eta}{\epsilon v_A}.$$

But

$$\begin{aligned}
l\omega_p &< \left(\frac{m_i}{m_e}\right)^{1/2} c \\
\frac{\eta}{\epsilon v_A}\omega_p &< \left(\frac{m_i}{m_e}\right)^{1/2} c \\
\frac{1}{\epsilon}\frac{c^2}{4\pi\sigma}\left(\frac{4\pi e^2 N_e}{m_e}\right)^{1/2}\left(\frac{B^2}{4\pi N_e m_i}\right)^{-1/2} &< \left(\frac{m_i}{m_e}\right)^{1/2} c \\
\frac{ceN_e}{\sigma\epsilon B} &< 1 \\
\frac{B\sigma}{N_e} &> \frac{ce}{\epsilon} \qquad (2.21) \\
\frac{B\sigma}{N_e} &> 10^2 \quad ; [c(cm\ s^{-1}); e(esu)]
\end{aligned}$$

(2.22)

This result indicates that the combination of strong fields and low electron densities are conducive to the enhancement of the resistivity in the plasma. The theoretical conductivity for small current densities in an ionized gas is $\sigma = 10^7 T^{1/2}$ (Parker 1976), resulting in

$$\begin{aligned}
\frac{B10^7T^{1/2}}{N_e} &> \frac{ce}{\epsilon} \\
\frac{BT^{1/2}}{N_e} &> \frac{ce}{10^7\epsilon}
\end{aligned} \qquad (2.23)$$

If these conditions are satisfied in a plasma the conductivity may be anomalously reduced in localized regions, resulting in particle acceleration.

The acceleration of particles through reduced conductivity in a highly conducting plasma is a very attractive mechanism to explain the non-thermal radio synchrotron outbursts from some astrophysical sources, like the Sun (during flares), quasars, radio stars and also AE Aquarii. The generation of strong electric fields in a plasma can lead to a phenomenon called "electron runaway acceleration", resulting in huge populations of electrons to be energized in a single "explosive" impulsive event (see e.g Benz 1994 for a detailed discussion). This occurs if the electric field generated in the plasma becomes greater than the so-called Dreicer field (Dreicer 1959), i.e. $E_D \sim 2 \times 10^{-10}$ (N_e/T) statvolt.cm^{-1}. For typical parameter values the Dreicer field is

$$E_D \approx 6 \times 10^{-3}\left(\frac{N}{10^9\,cm^{-3}}\right)\left(\frac{T}{10^4\,K}\right)^{-1} \text{Volt cm}^{-1} \qquad (2.24)$$

For all electric fields greater than the Dreicer field, the thermal electrons with velocities greater than the drift velocity of electrons relative to the heavier ions (Meintjes & de Jager 2000 for a detailed discussion), i.e. the high energy thermal tail of the MB distribution, will accelerate freely, experiencing virtually no friction as a result of ion-electron collisions. This will result in the whole high velocity thermal tail of the MB electron population to accelerate at once. The effectiveness of the acceleration process is then determined by the physical size of the medium, as well as energy loss processes of the electrons, like synchrotron emission and adiabatic losses.

2.4 Magnetic reconnection: Neutral sheet acceleration

The Sun's ever changing surface magnetism as a result of the continuous emergence of magnetic flux, generated by an internal dynamo mechanism, results in unremitting unrest above the Sun's surface. The reconnection of the newly immerged flux tubes with the existing flux tubes above the surface releases stored magnetic energy on an awesome scale, resulting in Solar flares. The amount of energy released through this process (i.e. magnetic reconnection) above the Sun's surface is equivalent to billions of thermonuclear explosions, raising the local temperature of Earth size regions in the Sun's atmosphere by tens of millions of degrees. Although the flares appear rather inconspicuous in visible light, they release enough energy in the X-ray and radio, through non-thermal radio synchrotron emission of vast amounts of relativistic electrons accelerated by electric fields and shocks in the acceleration zone (neutral sheet), to outshine the entire visible Sun.

Magnetic reconnection as the driving mechanism behind Solar flares and accompanying particle acceleration has been proposed by Parker (1957), refined by Petschek (1964) and has been extensively reviewed (Parker 1976; Benz 1994; Priest & Forbes 2000). The principle upon which the mechanism is based is the diffusion of magnetic regions of opposite polarity towards one another through a highly conducting plasma. The annihilation of magnetic flux in the merger will eventually be converted to kinetic and thermal energy of the plasma in the reconnection zone (see also Meintjes & de Jager 2000). The discussion that follows will illuminate only the basic principles of the reconnection process and acceleration, and is based on a general discussion of the Petschek model (Parker 1976).

Consider a magnetic field $\mathbf{B} = B(x)\hat{e}_z$ which is an odd function of x. The field approaches a uniform value $\pm B_o$ for large x (see Figure 9).

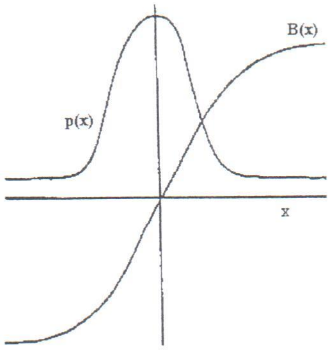

Figure 9 : Magnetic field strength and gas pressure with variation in x. (Parker 1976)

If hydrostatic equilibrium is to prevail in the x-direction the combination of the gas and field pressure must be uniform to avoid pressure gradients from forming, or

$$P(x) + \frac{B^2(x)}{8\pi} = P_\circ \quad \text{(constant)} \tag{2.25}$$

Since B(x = 0) = 0, the gas pressure P(x) is a maximum at x = 0 to support the whole configuration. The result is that there is nothing to counteract the high pressure at x = 0, preventing the gas from flowing away. This configuration is illustrated in Figure 9.

Two scenarios are now possible. The gas can flow along the lines of force and escape at the edge of the field or in blobs through the B-field due to large scale magnetic instabilities. If the gas escapes, the regions of opposite field come closer together while the condition for hydrostatic equilibrium may still be satisfied. The escape can continue and as a result the field gradient becomes steeper and steeper. Now if the transition layer between the two opposite fields has thickness $2l$, Ampère's law gives the current density as

$$\begin{aligned} 4\pi\mathbf{J} &= c\nabla\times\mathbf{B} \\ \mathbf{J} &= \frac{c}{4\pi}\left[\frac{dB(x)}{dy}\mathbf{i} - \frac{dB(x)}{dx}\mathbf{j} + 0\mathbf{k}\right] \\ &= -\frac{c}{4\pi}\frac{dB(x)}{dx}\mathbf{j} \end{aligned} \tag{2.26}$$

Then

$$J = -\frac{c}{4\pi}\frac{dB(x)}{dx} = O(\frac{cB(x)}{4\pi l}) \quad \text{; [in the y-direction]} \tag{2.27}$$

We see that the current density increases as l decreases. The presence of a current flowing in the y-direction points to the presence of an electric field in that direction. This field is given by

$$\mathbf{E} = -\frac{\mathbf{v}}{c}\times\mathbf{B} + \frac{c}{4\pi\sigma}\nabla\times\mathbf{B}$$

and the E-field must be in the y-direction :

$$
\begin{aligned}
E\mathrm{j} &= -\left(-\frac{\mathrm{v(x)B(x)}}{\mathrm{c}}\right)\mathrm{j} - \frac{\mathrm{c}}{4\pi\sigma}\frac{dB(x)}{dx}\mathrm{j} \\
&= -\frac{\mathrm{c}}{4\pi\sigma}\frac{d\mathrm{B}(x)}{dx}\mathrm{j} + \frac{\mathrm{v(x)B(x)}}{\mathrm{c}}\mathrm{j} \qquad (2.28)
\end{aligned}
$$

$$
\Rightarrow \quad E = \frac{\eta}{\mathrm{c}}\frac{d\mathrm{B}(x)}{dx} + \frac{\mathrm{v(x)B(x)}}{\mathrm{c}} \quad ; \text{[in the y-direction]} \qquad (2.29)
$$

We can imagine steady conditions where the B-field and the gas is convected in from each side at a velocity v(x) as rapidly as it is expelled and dissipated (by reconnection) near x = 0. This would mean that $\frac{d\mathrm{B}}{dt} = 0$ and hence from Faraday's induction law $\nabla \times \mathbf{E} = -\frac{1}{\mathrm{c}}\frac{\partial \mathrm{B}}{\partial t}$, we see that $\nabla \times \mathbf{E} = 0$.

Furthermore since $\mathrm{E} \equiv \mathrm{E}_y$, E is independent of x and z.

$$
\nabla \times \mathbf{E} = -\frac{dE_y}{dz}\mathbf{i} - 0\mathrm{j} + \frac{dE_y}{dx}\mathbf{k} = 0\mathrm{i} - 0\mathrm{j} + 0\mathbf{k}
$$

At x = 0 we get

$$
E = -\frac{\eta}{\mathrm{c}}\left(\frac{d\mathrm{B}}{dx}\right)_\circ \quad ; \text{[in the y-direction; } \mathrm{B}(x=0) = 0] \qquad (2.30)
$$

For large x, $\mathrm{B} \sim \mathrm{B}_\circ$ and thus we have $(\frac{d\mathrm{B}}{dx}) = 0$ and

$$
E = \frac{\mathrm{v(x)B(x)}}{\mathrm{c}} = \frac{\mathrm{v(x)B_\circ}}{\mathrm{c}} \qquad (2.31)
$$

or

$$
v(x) = \frac{\mathrm{c}E}{\mathrm{B}_\circ} = \frac{\mathrm{c}}{\mathrm{B}_\circ}\left(\frac{-\eta}{\mathrm{c}}\left(\frac{d\mathrm{B}}{dx}\right)_\circ\right) = -\frac{\mathrm{c}^2/4\pi\sigma}{\mathrm{B}_\circ}\left(\frac{d\mathrm{B}}{dx}\right)_\circ,
$$

assuming a constant E-field in the whole region.

This indicates that a particle may attain a velocity v(x) depending on the gradient in the magnetic field $\frac{d\mathrm{B}}{dx}$, near the position x = 0. The electric field E in the y-direction may accelerate particles to high energies depending on the field's strength. The creation and maintenance of such a field in a highly conducting fluid is however a difficult task. The best scenario is a region where the opposite B-fields come into contact over a very narrow region forming a steep gradient in the process.

From above (Eq.2.30) we have

$$
E = -\frac{\eta}{\mathrm{c}}\left(\frac{d\mathrm{B}}{dx}\right)_\circ = -\frac{\mathrm{c}}{4\pi\sigma}\left(\frac{d\mathrm{B}}{dx}\right)_\circ
$$

and therefore if σ is large, we need $\left(\frac{d\mathrm{B}}{dx}\right)_\circ \to \infty$ to get E to be large. This scenario of the annihilation of the magnetic field over a small region ($\Delta\mathrm{x} \to 0$) is known as the Petschek mechanism.

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The short region of contact causes the gas to squeeze out rapidly from between the fields. The scale length l may adjust to a value at which the fields near each other at $v_m = \epsilon v_A$ (merging speed) where $v_A = \frac{B}{\sqrt{4\pi\rho}}$ is the Alfvén speed of the medium, with $\epsilon = 0.1$ (e.g. Parker 1976; Priest 1981). The electric field generated in the merging region (MR) is then of the order of

$$E_{MR} = \frac{v B_\circ}{c}, = \frac{\epsilon v_A B_\circ}{c} \tag{2.32}$$

The energies particles can achieve with this mechanism depend on the length scale of the reconnection zone, which is still an unsolved theoretical dilemma. There is an alternative point of view regarding the actual acceleration of particles through magnetic reconnection (e.g Biskamp 1988; Lesch 1991), following directly from Faraday's induction law. These authors proposed that the actual site for particle acceleration is not the region where the merging fields approach one another, i.e. the merging region (MR), but the dissipation region (DR) where current flows along the field lines. In terms of this alternative description, it is proposed that a charged particle spiralling in the field in the MR will experience that the magnetic flux ($\Phi_m = B\pi R_L^2$) that passes through a hypothetical surface bounded by its orbit with Larmor radius R_L, will decrease at a rate $d\Phi_m/dt \sim -(B\pi R_L^2/\tau_r)$, where $\tau_r = R_L/v_m$. The work done per unit charge (*emf*) in one single orbit is (Biskamp 1988)

$$emf = 2\pi R_L E_{DR} = \frac{2\pi R_L \epsilon v_A B_\circ}{c} \tag{2.33}$$

where $E_{DR} = (1/\sigma)$ J is the accompanying electric field driving the current along the field lines away from the MR, with σ the conductivity of the plasma along the field. Effective particle acceleration will only occur if micro-instabilities are induced along the field as a result of the current, reducing σ to anomalously low values.

It is however generally accepted that magnetic reconnection is the mechanism behind the non-thermal outbursts on the Sun, as well as the particle acceleration in the geomagnetic tail. In the case of particle acceleration in the geomagnetic tail, for typical values of the magnetic field and particle density, i.e. $B_\circ = 2 \times 10^{-4}$ G and $n \approx 1$ cm^{-3} or $\rho \approx 6.67 \times 10^{-24}$ gcm^{-3}, it results in $v_A \approx 2.5 \times 10^7$ cm/s and $E \sim 10^{-9}$ statvolt/cm $= 5 \times 10^{-7}$ Volt/cm. If the length scale of acceleration is taken to be $\Delta L = 10^{11}$ cm, it can be shown that particles can be accelerated to energies like

$$\epsilon = e\Delta V \approx eE\Delta L \approx 10^4 \, \text{eV}$$

In Solar flares much higher energies can be achieved. The magnetic field in prominences is of the order of B $\approx 10^4$ Gauss, with typical particle densities of n $\approx 10^{11}$ cm^{-3} ($v_A \approx 7 \times 10^8$ cm s^{-1}). If acceleration can occur over length scales $\Delta L = 10^8$ cm, it can be shown that particles can be accelerated to energies of

$$\epsilon = e\Delta V \approx eE\Delta L \approx 10^{10}\,\mathrm{eV}$$

in these events. If a large population of run-away electrons with energies of the order of $\epsilon \approx 10$ GeV are produced in a flare event, it will explain the non-thermal radio synchrotron outbursts during Solar flares and this mechanism may well be applicable to AE Aquarii.

2.5 Double-layer formation and particle acceleration

The acceleration of particles along field lines (e.g in the DR) are closely linked to the formation of so-called double layers. A qualitative description of double layer formation will be presented next.

Consider a magnetic field **B** embedded in a highly conducting fluid. Let the fluid motion be along $\hat{e}_x$ for all z > h but no fluid motion for z < 0. The velocity is therefore $\mathbf{v} = v_x\hat{e}_x$. This motion has the effect of moving charges across the B-field (we consider a field in the z-direction, $\mathbf{B} = B_z\hat{e}_z$), resulting in an ϵmf which can drive a current. In a highly conducting fluid this is given by (as from Eq.2.4)

$$\mathbf{E} = -\frac{1}{c}\mathbf{v}\times\mathbf{B}$$

or

$$\begin{aligned}\mathbf{E} &= -\frac{1}{c}(v_x\hat{e}_x \times B_z\hat{e}_z)\\ &= -\frac{v_x B_z}{c}(-\hat{e}_y)\\ &= E_y\hat{e}_y \qquad ;[E_y = \frac{v_x B_z}{c} = vB/c]\end{aligned} \tag{2.34}$$

In the gap (0 < z < h) we have

$$\mathbf{E} = -\nabla\phi$$

This electric field will drive a current at z > h.

But first

$$\begin{aligned}\mathbf{E} &= -\nabla\phi\\ (E_x\hat{e}_x + E_y\hat{e}_y + E_z\hat{e}_z) &= -\left(\frac{d\phi}{dx}\hat{e}_x + \frac{d\phi}{dy}\hat{e}_y + \frac{d\phi}{dz}\hat{e}_z\right)\end{aligned}$$

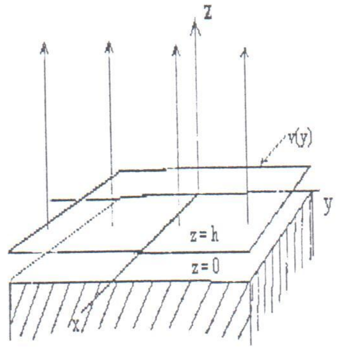

Figure 16 : A visualization of a double-layer. (Parker 1976)

$$\text{but} \quad E_x, E_z = 0 \;\; (z > h)$$

$$\mathbf{E} = \mathbf{E_y \hat{e}_y} = -\frac{d\phi}{dy}\hat{e}_y \tag{2.35}$$

So at z = h,

$$E_y = -\frac{\Delta\phi}{\Delta y} = -\left[\frac{\phi(y,h) - \phi(y=0,h)}{\Delta y}\right]$$

$$\begin{aligned} -\nabla\phi &= E_y \Delta y \\ -[\phi(y,h) - \phi(y=0,h)] &= \int_0^y E_y dy \\ \Rightarrow \quad \phi(y,h) &= \phi(y=0,h) - \int_0^y E_y dy \quad ;[\text{on } z = h] \end{aligned} \tag{2.36}$$

The scenario as described above is depicted in Figure 16.

Now for 0 < z < h

$$\begin{aligned} z = 0 \;\; ;E_y &= 0 \\ z = h \;\; ;E_y &= -\frac{\Delta\phi}{\Delta y} \end{aligned}$$

We also have

$$E_z = -\nabla\phi_{,z} = \frac{\partial\phi}{\partial z}\hat{e}_z$$

but we see that

$$E_z = -\frac{\Delta\phi}{\Delta z} \approx O\left(\frac{\phi}{h}\right) \tag{2.37}$$

and

$$E_y = -\frac{\Delta\phi}{\Delta y} \approx O\left(\frac{\phi}{l}\right) \tag{2.38}$$

where l is a length along the y-axis.

Therefore for $l \gg$ h

$$E_y \ll E_z.$$

The boundary conditions are now

$$z = 0 \quad ; \quad \phi(y, z = 0) = 0 \tag{2.39}$$

$$z = h \quad ; \quad \phi(y, h) = \phi(y = 0, h) - \int_0^y E_y dy. \tag{2.40}$$

In the z-direction we have

$$\begin{aligned}
\Delta\phi_{,z} &= \phi(y, z = h) - \phi(y, z = 0) \\
&= [\phi(y = 0, h) - \int_0^y E_y dy] - \phi(y, z = 0) \\
&= \phi(y = 0, h) - \int_0^y E_y dy \\
&\approx -\int_0^y E_y dy.
\end{aligned} \tag{2.41}$$

Then in the gap we get

$$\begin{aligned}
E_z &= -\frac{\Delta\phi}{\Delta z} \\
&\approx \frac{\int_0^l E_y dy}{h} \\
&\approx \frac{\frac{\mathrm{vB}}{\mathrm{c}} \int_0^l dy}{h} \\
&= E_y \frac{l}{h}
\end{aligned} \tag{2.42}$$

and then for z = h

$$E_y = \mathrm{vB/c}$$

and for z = 0

$$E_y = 0.$$

For $0 < z < h$:

$$E_y(z) = \int_0^z \frac{E_y(z = h)}{h} dz = E_y \frac{z}{h}. \tag{2.43}$$

Thus in summary :

$$\begin{aligned}
E_z &\approx E_y \frac{l}{h} \\
E_y(z) &\approx E_y \frac{z}{h}.
\end{aligned}$$

In the gap we have

$$E_z(z) \gg E_y(z) \quad \text{for} \quad l \gg z.$$

Figure 17 gives an idea of the electric field configuration in the case of a double layer.

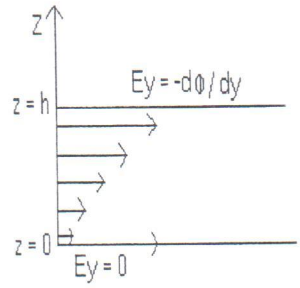

Figure 17 : The variation of the electric field in the gap.

One can see from this result that currents flowing in the B-field direction (z-direction) are generated outside the zone of cross-field motion.

$$E_{\|} = E_z = E_y(z \geq h)\frac{l}{h}$$
$$E_{\perp} = E_y(z \geq h)\frac{z}{h}$$

where $\mathrm{E}_y = \mathrm{vB/c}$. The electric fields generated through this process can result in particle acceleration, if the conductivity is reduced anomalously through the generation of micro-instabilities.

2.5.1 The electrostatic diode

A more detailed discussion can be given of a double layer by considering the example of the electrostatic diode. Let such a structure be between z = a and z = b. We consider the parameters $\mathrm{m}_{i,e}$, $\mathrm{u}_{i,e}(z)$ and $N_{i,e}(z)$ which represent for ions or electrons the mass, velocity and number density along the z-axis, respectively.

The flux of ions in the positive z-direction (upwards) is

$$\Phi(z) = \mathrm{N}_i(z)u_i(z) \quad (\mathrm{cm}^{-2}\mathrm{s}^{-1}) \tag{2.44}$$

and similar for the electron flux, only in the opposite direction

$$\Phi(z) = \mathrm{N}_e(z)u_e(z) \quad (\mathrm{cm}^{-2}\mathrm{s}^{-1}). \tag{2.45}$$

The total charge density then is

$$\begin{aligned}\rho(z) &= \mathrm{N}_i(z)\mathrm{q}_+ + \mathrm{N}_e(z)\mathrm{q}_- \\ &= [\mathrm{N}_i(z) - \mathrm{N}_e(z)]e.\end{aligned} \tag{2.46}$$

This leads to a current density of

$$\begin{aligned} \mathrm{J}(z) &= \mathrm{q}_+\mathrm{N}_i(z)u_i(z) + \mathrm{q}_-\mathrm{N}_e(z)u_e(z) \\ &= [\Phi_i(z) - \Phi_e(z)]e \qquad (\text{statampere cm}^{-2}). \end{aligned} \tag{2.47}$$

If steady conditions are to prevail, Φ_i and Φ_e must be uniform over z and t. If this was not the case, charges would accumulate in different regions and electric fields would start currents flowing to reduce the imbalances.

The vertical electric field relates to the potential ϕ like

$$E(z) = -\frac{d\phi}{dz}.$$

The potential must also satisfy Poisson's equation.

$$\nabla^2\phi = -4\pi\rho. \tag{2.48}$$

If we ignore x and y dependence this gives

$$\frac{d^2\phi}{dz^2} = -4\pi\rho(z) = -4\pi(\mathrm{N}_i(z) - \mathrm{N}_e(z))e. \tag{2.49}$$

We now let ϕ vanish at z = b and let $\phi = \phi_\circ$ at z = a. As from above the ion flux is upwards and thus the electron flux is downwards. We consider negligible velocities for the ions and electrons as they enter the gap (a < z < b), ions from below and electrons from above. Therefore the velocities can be found as a function of z by considering the work-energy theorem

$$\mathrm{W}_{\text{field}} = \Delta\mathrm{KE}_{\text{particle}}.$$

This then gives

$$\begin{aligned} \mathrm{W} &= \Delta(\mathrm{F}_e\mathrm{z}) \\ &= \Delta(\mathrm{qEz}) = \mathrm{q}\Delta(\mathrm{Ez}) = \mathrm{q}\Delta\phi. \end{aligned} \tag{2.50}$$

For protons (ions) we have

$$\Delta\phi(z) = \phi_\circ - \phi(z) > 0 \quad [\text{protons gain energy in the gap}]$$

and therefore

$$\begin{aligned} \mathrm{W} = e(\phi_\circ - \phi(z)) &= 1/2\mathrm{m}_i u_i^2(z) - 1/2\mathrm{m}_i u_i^2\ (z=a) \\ &= 1/2\mathrm{m}_i u_i^2(z). \end{aligned} \tag{2.51}$$

And also for electrons

$$\Delta\phi(z) = \phi(z = b) - \phi(z) = -\phi(z)$$

and then to gain energy (choosing the sign correctly) in the gap

$$\begin{aligned} \mathrm{W} = -e(\Delta\phi(z)) &= 1/2\mathrm{m}_e u_e^2(z) - 1/2\mathrm{m}_e u_e^2\ (z = b) \\ &= 1/2\mathrm{m}_e u_e^2(z) \end{aligned} \tag{2.52}$$

and therefore

$$e\phi(z) = 1/2\mathrm{m}_e u_e^2(z). \tag{2.53}$$

In summary then

$$\frac{1}{2}\mathrm{m}_i u_i^2(z) = e(\phi_\circ - \phi(z)) \tag{2.54}$$

$$\frac{1}{2}\mathrm{m}_e u_e^2(z) = e\phi(z)). \tag{2.55}$$

Now we can solve for the respective velocities.

$$u_i^2 = \frac{2e(\phi_\circ - \phi(z))}{\mathrm{m}_i} \tag{2.56}$$

$$u_e^2 = \frac{2e\phi(z)}{\mathrm{m}_e}. \tag{2.57}$$

We proceed with substitution into Poisson's equation as goal in mind.

Then from Eq.2.44 the respective densities for ions and electrons are

$$\mathrm{N}_i(z) = \frac{\Phi_i(z)}{u_i(z)} \quad ; \quad \mathrm{N}_e(z) = \frac{\Phi_e(z)}{u_e(z)}.$$

Therefore

$$\mathrm{N}_i(z) = \Phi_i(z)\left[\frac{\mathrm{m}_i}{2e(\phi_\circ - \phi(z))}\right]^{1/2} \tag{2.58}$$

$$\mathrm{N}_e(z) = \Phi_e(z)\left[\frac{\mathrm{m}_e}{2e\phi(z)}\right]^{1/2}. \tag{2.59}$$

This can be substituted into Poisson's equation

$$\begin{aligned} \frac{d^2\phi}{dz^2} &= -4\pi e(\mathrm{N}_i(z) - \mathrm{N}_e(z)) \\ &= -4\pi e\left[\Phi_i(z)\left(\frac{\mathrm{m}_i}{2e(\phi_\circ - \phi(z))}\right)^{1/2} - \Phi_e(z)\left[\frac{\mathrm{m}_e}{2e\phi(z)}\right]^{1/2}\right] \\ &= 4\pi e\left[\Phi_e(z)\left(\frac{\mathrm{m}_e}{2e\phi(z)}\right)^{1/2} - \Phi_i(z)\left(\frac{\mathrm{m}_i}{2e(\phi_\circ - \phi(z))}\right)^{1/2}\right]. \end{aligned} \tag{2.60}$$

Now consider the following

$$\frac{d}{dz}\left(\frac{d\phi}{dz}\right)^2 = 2\frac{d\phi}{dz}\frac{d^2\phi}{dz^2}$$

or

$$\Rightarrow \quad \frac{d^2\phi}{dz^2} = \frac{1}{2}\frac{dz}{d\phi}\frac{d}{dz}\left(\frac{d\phi}{dz}\right)^2 = \frac{1}{2}\frac{d}{d\phi}\left(\frac{d\phi}{dz}\right)^2.$$

So

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\phi}\left(\frac{d\phi}{dz}\right)^2 &= 4\pi e\left[\Phi_e(z)\left(\frac{\mathrm{m}_e}{2e\phi(z)}\right)^{1/2} - \Phi_i(z)\left(\frac{\mathrm{m}_i}{2e(\phi_\circ-\phi(z))}\right)^{1/2}\right] \\
\int d\left(\frac{d\phi}{dz}\right)^2 &= \frac{8\pi e}{(2e)^{1/2}}\left[\Phi_e(z)\mathrm{m}_e^{1/2}\int\frac{d\phi}{\phi^{1/2}(z)} - \Phi_i(z)\mathrm{m}_i^{1/2}\int\frac{d\phi}{(\phi_\circ-\phi(z))^{1/2}}\right] \\
\left(\frac{d\phi}{dz}\right)^2 &= \frac{8\pi e}{(2e)^{1/2}}\left[\Phi_e(z)\mathrm{m}_e^{1/2}\frac{\phi^{1/2}(z)}{1/2} - \Phi_i(z)\mathrm{m}_i^{1/2}\int\frac{d\phi}{(-\phi(z)+\phi_\circ)^{1/2}}\right].
\end{aligned} \tag{2.61}$$

The second integral has a standard solution like $\int\frac{dx}{(ax+b)^{1/2}} = \frac{2\sqrt{ax+b}}{a}$, which in our case is = $-2(\phi_\circ-\phi(z))^{1/2}$.

Therefore

$$\begin{aligned}
\left(\frac{d\phi}{dz}\right)^2 &= \frac{8\pi e}{(2e)^{1/2}}\left[2\Phi_e(z)\mathrm{m}_e^{1/2}\phi^{1/2}(z) + 2\Phi_i(z)\mathrm{m}_i^{1/2}(\phi_\circ-\phi(z))^{1/2}\right] \\
&= 8\pi(2e)^{1/2}\left[\Phi_e(z)\mathrm{m}_e^{1/2}\phi^{1/2}(z) + \Phi_i(z)\mathrm{m}_i^{1/2}(\phi_\circ-\phi(z))^{1/2}\right].
\end{aligned} \tag{2.62}$$

If the ion mass is very high implying $\mathrm{u}_i \ll \mathrm{u}_e$ then

$$\begin{aligned}
\mathrm{m}_i^{1/2}\Phi_i(z) &\ll \mathrm{m}_e^{1/2}\Phi_e(z) \\
\mathrm{m}_i^{1/2}\mathrm{N}_i(z)u_i(z) &\ll \mathrm{m}_e^{1/2}\mathrm{N}_e(z)u_e(z) \\
\Rightarrow \quad u_i(z) &\ll u_e(z).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{d\phi}{dz}\right)^2 &\approx 8\pi(2e)^{1/2}(\Phi_e(z)\mathrm{m}_e^{1/2}\phi^{1/2}(z)) \\
\frac{d\phi}{dz} &= \pm\left[8\pi(2e)^{1/2}(\Phi_e(z)\mathrm{m}_e^{1/2}\phi^{1/2}(z))\right]^{1/2} \\
&= \pm\left[128\pi^2 e\Phi_e^2(z)\mathrm{m}_e\phi(z)\right]^{1/4}.
\end{aligned} \tag{2.63}$$

Then

$$\int_0^{\phi(z)}\phi^{-1/4}(z)d\phi = -\left[128\pi^2 e\Phi_e^2(z)\mathrm{m}_e\right]^{1/4}\int_b^z dz \quad ;\ \text{[for a positive solution]}$$

$$\begin{aligned}\phi(z) &= (3/4)^{4/3}\left[128\pi^2 e\Phi_e^2(z)\mathrm{m}_e\right]^{1/3}(b-z)^{4/3}\\ &= \left[(81/2)\pi^2 e\Phi_e^2(z)\mathrm{m}_e(b-z)^4\right]^{1/3}\end{aligned}$$

resulting in

$$\begin{aligned}\phi(z) &= \left[\frac{81\pi^2}{2e}e^2\Phi_e^2(z)\mathrm{m}_e(b-z)^4\right]^{1/3}\\ &= \left(\frac{81\pi^2\mathrm{m}_e}{2e}\right)^{1/3}(e\Phi_e(z))^{2/3}(b-z)^{4/3}\\ &= 10^{-5}\mathrm{J}_e^{2/3}(z)(b-z)^{4/3}\quad\text{(statvolt)}\end{aligned}$$

$$\Rightarrow\quad \phi(z) = 3\times10^{-3}\mathrm{J}_e^{2/3}(z)(b-z)^{4/3}\quad\text{(Volt)}\quad[1\,\text{statvolt}=300\,\text{Volt}]. \tag{2.64}$$

One can illustrate this at the hand of double layer formation in prominences on the Sun during eruptions. For typical values we get from

$$\phi(z) = 3\times10^{-3}\mathrm{J}_e^{2/3}(z)(b-a)^{4/3}\ \text{Volt},$$

and for $\mathrm{J}\approx\frac{c\mathrm{B}}{4\pi l}\approx 2400\left(\frac{\mathrm{B}}{10\mathrm{G}}\right)\left(\frac{l}{10^7\mathrm{cm}}\right)^{-1}$ esu, it results in a potential difference of the order of

$$\phi(\text{gap}) \approx 1\,\text{GigaVolt}\left(\frac{\mathrm{J}_e}{2400\text{esu}}\right)^{2/3}\left(\frac{b-a}{10^7\text{cm}}\right)^{4/3}$$

and particle energies of

$$E_e = e\phi(\text{gap}) \approx 1\,\text{GeV}.$$

This mechanism therefore has the potential to accelerate electrons to high energies during Solar flares, explaining the non-thermal emission during these events.

2.6 Adiabatic compression in magnetic fields: Betatron Acceleration

Betatron acceleration is widely considered as the acceleration mechanism responsible for the acceleration of charged particles in the Earth's magnetosphere, as well as the mechanism responsible for the energy gain of relativistic electrons during their interaction with supernova envelopes at the later stages of the evolution of the envelope. The betatron process is based upon the conservation of the adiabatic invariant of a charged particle ($\mu = \mathrm{p}_\perp^2/\mathrm{B} = \text{const}$) when moving in a magnetic field of varying intensity. In this expression $\mathrm{p}_\perp = \mathrm{m}_p\ \mathrm{w}_\perp$ represents the component of momentum of the particle perpendicular to the field, and B the magnetic intensity. A discussion of this process will now be given.

Consider n(cm^{-3}) particles with particle mass m_p(g) and mean square velocity $< u^2 > = u^2$ in a certain direction. We calculate the pressure exerted by the particle in that direction. The pressure generally is

$$\begin{aligned} P &= 1/3n < \mathbf{p}.\mathbf{v} > = 1/3n < m_p \mathbf{v}.\mathbf{v} > \\ &= 1/3n m_p < v^2 > \end{aligned} \tag{2.65}$$

but (for example) $< v^2 > = < v_x^2 + v_y^2 + v_z^2 > = < 3v_x^2 > = 3 < v_x^2 >$.

So in one dimension :

$$P = 1/3n m_p.3 < v_x^2 > = n m_p < v_x^2 > . \tag{2.66}$$

Thus

$$\Rightarrow \quad P = n m_p < u^2 > = n m_p u^2.$$

The first law of thermodynamics for adiabatic variations of a volume V is

$$dU + PdV = 0 \tag{2.67}$$

where U is the kinetic energy contained in the volume V. Using the expression for energy and pressure, this results in

$$d[(1/2n m_p u^2)V] + (n m_p u^2)dV = 0.$$

But $d(uv) = vdu + udv$

$$\begin{aligned} Vd(1/2n m_p u^2) + 1/2n m_p u^2 dV + (n m_p u^2)dV &= 0 \\ \frac{d(1/2n m_p u^2)}{(1/2n m_p u^2)} + 3\frac{dV}{V} &= 0 \\ 3\int \frac{dV}{V} + \int \frac{d(1/2n m_p u^2)}{(1/2n m_p u^2)} &= \ln(\text{const}) \\ 3\ln V + \ln(1/2n m_p u^2) &= \ln(\text{const}) \\ \frac{1}{2} n m_p u^2 V^3 &= \text{const.} \end{aligned} \tag{2.68}$$

But

$$\frac{1}{2} m_p = \text{const}$$

$$\Rightarrow \quad n u^2 V^3 = \text{const.}$$

If the number of particles is conserved

$$N = n(\text{cm}^{-3})V(\text{cm}^3) = \text{const}$$

then the equation becomes

$$\begin{aligned} (nV)(u^2V^2) &= \text{const} \\ \rightarrow \quad (uV)^2 &= \text{const} \end{aligned} \tag{2.69}$$

or

$$uV = \text{const.} \tag{2.70}$$

When considering one-dimensional compression V can be replaced by l and thus

$$ul = \text{const}$$

where u is for motion parallel to l.

This applies to compression in the absence of a B-field and for compression parallel to a field since the motion of charged particles are confined by the field in the perpendicular direction. Now if the dimension l is confined (compression) then the velocity has to adjust.

$$\Rightarrow \quad l \rightarrow 0$$

$$\Rightarrow \quad u \rightarrow \infty \quad \text{(particles accelerate).}$$

The motions perpendicular to the magnetic field (x and y directions) are strongly coupled by the gyrations around field lines. This must be taken into account for the case of compression perpendicular to the magnetic field. We can consider the mean square particle velocity in either of the two directions to find the kinetic energy density.

Per particle we have :

(Let $\mathbf{B} = \mathrm{B}\hat{e}_z$)

$$\begin{aligned} \mathrm{KE} &= \frac{1}{2}\mathrm{m}_p < v_x^2 + v_y^2 > \\ &= \frac{1}{2}\mathrm{m}_p[< v_x^2 > + < v_y^2 >] \\ &\approx \mathrm{m}_p < v_x^2 > \quad \text{[in direction of motion]} \end{aligned}$$

$$\Rightarrow \quad \mathrm{KE} \approx \mathrm{m}_p u^2. \tag{2.71}$$

The kinetic energy density then is

$$U = n\mathrm{m}_p u^2. \tag{2.72}$$

Applying the first law of thermodynamics to this example gives

$$
\begin{aligned}
\mathrm{dU} + \mathrm{PdV} &= 0 \\
d(n\mathrm{m}_p u^2 V) + (n\mathrm{m}_p u^2)dV &= 0
\end{aligned}
$$

like before :

$$
\begin{aligned}
Vd(n\mathrm{m}_p u^2) + 2n\mathrm{m}_p u^2)dV &= 0 \\
\int \frac{d(n\mathrm{m}_p u^2)}{n\mathrm{m}_p u^2} + 2\int \frac{dV}{V} &= \ln(\mathrm{const}) \\
\Rightarrow \quad nu^2V^2 &= \mathrm{const} \\
\Rightarrow \quad u^2V &= \mathrm{const} \quad ; (N = nV \text{ is conserved}). \qquad (2.73)
\end{aligned}
$$

In terms of a two-dimensional motion with $\mathrm{V} \propto l^2$

$$a) \quad \Rightarrow \quad u^2 l^2 = \mathrm{const.}$$

But the conservation of magnetic flux also requires that

$$b) \quad \Rightarrow \quad \mathrm{B}l^2 = \mathrm{const}$$

or

$$l^2 = \frac{\mathrm{const}}{\mathrm{B}}.$$

This is put into $a)$ to give

$$\frac{u^2}{\mathrm{B}} = \mathrm{const.} \qquad (2.74)$$

Usually the r.m.s velocity for the two directions $\perp$ to the field is written as $\mathrm{v}_\perp^2$. Therefore if $\mathbf{B} = \mathrm{B}\hat{\mathbf{e}}_\mathbf{z}$ then the x- and y-directions are the $\perp$ directions and

$$
\begin{aligned}
1/2 < v_x^2 + v_y^2 > &= 1/2 < v_\perp^2 > \\
&= 1/2(2 < u^2 >) = u^2. \qquad (2.75)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
< w_\perp^2 > &= 2u^2 \quad ; [\text{with definition : } < w_\perp^2 > = < u_x^2 + u_y^2 >] \\
w_\perp^2 &= 2u^2. \qquad (2.76)
\end{aligned}
$$

Then $\frac{u^2}{\mathrm{B}} = \mathrm{const}$

$$\Rightarrow \quad \frac{1/2w_\perp^2}{\mathrm{B}} = \mathrm{const.}$$

$$\Rightarrow \quad \frac{1/2\mathrm{m}_p w_\perp^2}{\mathrm{B}} = \text{const} \;\; ; \;\; [\mathrm{m}_p = \text{const}, \text{non-relativistic case}]. \tag{2.77}$$

Particles that move parallel to the B-field with speed $u_{||}$ between approaching magnetic mirrors are also compressed to give, in the one-dimensional case,

$$n(\mathrm{cm}^{-1})l = \text{const} \quad ; \; [\text{conservation of particles}]$$

where l is the separation between the mirrors.

For simultaneous transverse and longitudinal compression we can write for such a 3D compression

$$V(\mathrm{cm}^3)n(\mathrm{cm}^{-3}) = \text{const}$$

and

$$V = A(\mathrm{cm}^2)l.$$

This then gives

$$n(\mathrm{cm}^{-3})A(\mathrm{cm}^2)l = \text{const}. \tag{2.78}$$

Conservation of B-flux requires

$$\mathrm{B}A = \text{const}$$

$$\mathrm{B}l^2 = \text{const}.$$

Then from above

$$a)(\text{Eq.2.76}) \quad \frac{w_\perp^2}{\mathrm{B}} = \text{const}$$

$$b) \quad w_{||}l = \text{const} = \frac{w_\perp l^2}{l}.$$

Therefore from b) and Eq.2.77

$$\begin{aligned}
\frac{w_{||}Anl}{nl^2} &= \text{const} \quad ; \quad [An = nl^2] \\
\frac{w_{||}\text{const}}{nl^2} &= \text{const} \quad ; \quad [Anl = \text{const}] \\
\frac{w_{||}\mathrm{B}}{\mathrm{nB}l^2} &= \text{const} \\
\Rightarrow \quad \frac{w_{||}\mathrm{B}}{n} &= \text{const} \quad ; \quad [\mathrm{B}l^2 = \text{const}].
\end{aligned} \tag{2.79}$$

Hence in summary

$$\frac{w_\perp^2}{\mathrm{B}} = \text{const} \tag{2.80}$$

$$\frac{w_{||}\mathrm{B}}{n} = \text{const}. \tag{2.81}$$

The particle pressures are :

$$\mathrm{P}_\perp = \frac{1}{2} n \mathrm{m}_p w_\perp^2 \tag{2.82}$$

$$\mathrm{P}_\parallel = n \mathrm{m}_p w_\parallel^2. \tag{2.83}$$

Then

$$\begin{aligned} \frac{\mathrm{P}_\perp}{n\mathrm{B}} &= \frac{1/2 n \mathrm{m}_p w_\perp^2}{n\mathrm{B}} \\ &= \frac{1/2 \mathrm{m}_p w_\perp^2}{\mathrm{B}} = \text{const.} \end{aligned} \tag{2.84}$$

Also

$$\begin{aligned} \frac{\mathrm{P}_\parallel}{n\mathrm{B}} &= \frac{n \mathrm{m}_p w_\parallel^2}{n\mathrm{B}} \\ &= \frac{n^2 \mathrm{m}_p}{\mathrm{B}^3} \frac{w_\parallel^2 \mathrm{B}^2}{n^2} \\ &= \text{const}\frac{n^2 \mathrm{m}_p}{\mathrm{B}^3}. \end{aligned} \tag{2.85}$$

Then

$$\frac{\mathrm{B}^2 \mathrm{P}_\parallel}{\mathrm{m}_p n^3} = \text{const.}$$

Therefore in summary

$$\frac{\mathrm{P}_\parallel \mathrm{B}^2}{n^3} = \text{const} \tag{2.86}$$

$$\frac{\mathrm{P}_\perp}{\mathrm{nB}} = \text{const.} \tag{2.87}$$

We can therefore see that for magnetic compression of particles we get

$$(\mathrm{nB}) \to \infty \Rightarrow \mathrm{P}_\perp = 1/2 n \mathrm{m}_p w_\perp^2 \to \infty$$

resulting in

$$w_\perp^2 \to \infty.$$

The average KE of the particles increases to keep the ratio $\frac{\mathrm{P}_\perp}{\mathrm{nB}}$ constant. Furthermore, for parallel compression we get if

$$n^3 \to \infty$$

$$\mathrm{P}_\parallel \mathrm{B}^2 \to \infty$$

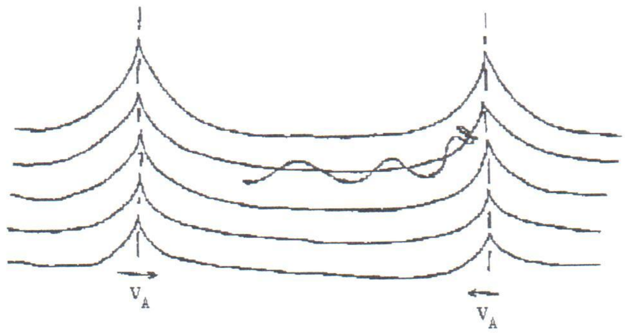

Figure 18 : A particle can be trapped between two approaching magnetic waves. (Parker 1976)

resulting in

$$
\begin{aligned}
n m_p w_{\|}^2 B^2 &\rightarrow \infty \\
\Rightarrow \quad w_{\|}^2 &\text{ will also increase.}
\end{aligned}
\tag{2.88}
$$

It must be kept in mind that this process is completely reversible, unless the particles can somehow leak out of the acceleration zone when they have higher energies, i.e. where the field gradient is largest. A specific example of the acceleration of particles by this process is the scattering of charged particles in a turbulent plasma from Alfvén waves, i.e.magnetic discontinuities propagating along field lines with a typical speed, the Alfvén speed. (Parker 1958; Parker 1976). A discussion of this process will now be presented.

Imagine a particle with pitch angle $\theta_\circ(\mathrm{t})$ in a field $B_\circ$. The particle is trapped between two approaching Alfvén waves with maximum field density $B_w > B_\circ$, where $B_\circ$ is the field strength in the interwave zone. The condition for the particle to be trapped in these approaching fields is given by $\sin^2\theta_\circ(t_\circ) > \frac{B_\circ}{B_w}$.

The particle is reflected back and forth between the two waves. A charge that approaches a wave crest 'tries' to follow the field line but gets flipped back in the process. A graphical representation is shown in Figure 18. The kinetic energy of the motion parallel to the field satisfies

$$
\frac{1/2 m_p w_\perp^2}{B} = \text{const} \tag{2.89}
$$

and this kinetic energy does not change between the waves or

$$
w_\perp(t) = w_\perp(0).
$$

For motion parallel to the field

$$
w_{\|}(t) l(t) = \text{const},
$$

where $l(t)$ is the separation between two waves.

The pitch angle θ is given by

$$\tan\theta = \frac{w_\perp}{w_\parallel}.$$

In the increased field of a wave the pitch angle increases from $\theta_\circ$ in the $B_\circ$ field to

$$\sin^2\theta(t) = \frac{B}{B_\circ}\sin^2\theta_\circ(t). \tag{2.90}$$

Considering the constant ratio (Eq.2.88), as $\mathbf{w_p}$ increases with successive reflections from head-on waves, the angle $\theta_\circ(t)$ (the angle in the uniform field between the waves) will decrease. This process will continue until $\theta(t)$ fails to reach $\pi/2$ which is the reflection angle. The field is $B = B_w$ at this flip back stage and therefore the critical pitch angle is reached for

$$\begin{aligned}\sin^2\theta_\circ(t) &= \frac{B_\circ}{B_w}\sin^2\theta(t)\\ &\le \frac{B_\circ}{B_w} \qquad ;\ [\sin^2\theta \le 1\ ;\theta \le \pi/2].\end{aligned} \tag{2.91}$$

At this stage the parallel velocity of the deflected particle is

$$\begin{aligned}\tan\theta_\circ(t) &= \frac{w_\perp}{w_\parallel}\\ w_\parallel^2 &= \frac{\cos^2\theta_\circ(t)}{\sin^2\theta_\circ(t)}w_\perp^2\\ &= w_\perp^2\frac{1-\sin^2\theta_\circ}{\sin^2\theta_\circ}\\ &= w_\perp^2\left[\frac{B_w}{B_\circ}-1\right].\end{aligned} \tag{2.92}$$

Also

$$\begin{aligned}w^2(t) &= w_\parallel^2(t) + w_\perp^2(t)\\ &= w_\perp^2(t)\left[\frac{B_w}{B_\circ}-1\right] + w_\perp^2(t)\\ &= w_\perp^2(t)\frac{B_w}{B_\circ} = w_\perp^2(t)[\cot^2\theta_\circ(t)+1].\end{aligned} \tag{2.93}$$

If the initial pitch angle was $\theta_\circ(t_\circ)$ when the particle entered the region between the waves, the critical velocity was

$$\begin{aligned}w^2(t_\circ) &= w_\parallel^2(t_\circ) + w_\perp^2(t_\circ)\\ &= w_\perp^2(t_\circ)\left[\frac{B_w}{B_\circ}-1\right] + w_\perp^2(t_\circ)\\ &= w_\perp^2(t_\circ)[\cot^2\theta_\circ(t_\circ)+1].\end{aligned} \tag{2.94}$$

The gain in KE before the particle escapes is therefore

$$
\begin{aligned}
\frac{w^2(t)}{w^2(t_\circ)} &= \frac{w_\perp^2(t)[\cot^2\theta_\circ(t)+1]}{w_\perp^2(t_\circ)[\cot^2\theta_\circ(t_\circ)+1]} \\
&= \frac{\frac{\cos^2\theta_\circ(t)+1}{\sin^2\theta_\circ(t)}+1}{\frac{\cos^2\theta_\circ(t_\circ)+1}{\sin^2\theta_\circ(t_\circ)}+1} \\
&= \frac{\mathrm{B}_w}{\mathrm{B}_\circ}\sin^2\theta_\circ(t_\circ). \qquad (2.95)
\end{aligned}
$$

If, for example, $\theta_\circ(t_\circ) = \pi/3$ and $\frac{\mathrm{B}_w}{\mathrm{B}_\circ} \approx 2$ then

$$\frac{w^2(t)}{w^2(t_\circ)} \approx 1.5$$

or the kinetic energy increases by 50 % before the particle escapes. However, if the particles that are trapped in the wave zone (between approaching crests) have their pitch angles continuously isotropized as a result of scattering off whistler waves and other particles, the energy gained before they tunnel out can be substantially greater. This process can play a very important role in the continuous acceleration of charged particles that are trapped inside magnetized gas that are continuously battered by a magnetospheric propeller, like proposed for AE Aquarii.

2.7 Fermi acceleration

The Fermi mechanism was first proposed by Fermi (1949) as a stochastic means by which particles colliding with clouds or inhomogeneities in the interstellar medium (ISM) could be accelerated to very high energies, accounting for the high energy cosmic rays observed in ground based and balloon experiments. In a collision of a particle with a moving inhomogeneity, the magnitude of the relative velocity remains unchanged. This is the same concept as the elastic collision between a particle and a moving piston.

Therefore

$$|v_{\mathrm{rel}}(\mathrm{before})| = |v_{\mathrm{rel}}(\mathrm{after})|.$$

Consider an overtaking type of collision (see Figure 19) :

$$v_{\mathrm{rel}}(\mathrm{before}) = w_p\cos\theta - u$$

$$v_{\mathrm{rel}}(\mathrm{after}) = w_p'\cos\theta' + u.$$

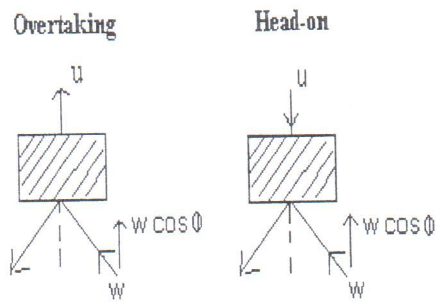

Figure 19 : The difference in the head-on and the overtaking types of collisions.

Then

$$
\begin{aligned}
w_p \cos\theta - u &= w'_p \cos\theta' + u \\
w'_p \cos\theta' &= w_p \cos\theta - 2u. \qquad (2.96)
\end{aligned}
$$

The other possibility is a head-on collision :

$$v_{\mathrm{rel}}(\text{before}) = w_p \cos\theta + u$$

$$v_{\mathrm{rel}}(\text{after}) = w'_p \cos\theta' - u.$$

Then

$$w'_p \cos\theta' = w_p \cos\theta + 2u. \qquad (2.97)$$

Now consider the one-dimensional case :

$$w'_p = w_p + 2u \quad \text{[head-on]}$$

$$w'_p = w_p - 2u \quad \text{[overtaking]}.$$

A particle in the fixed frame (its own stationary frame of reference) gains kinetic energy T for Head-on:

$$
\begin{aligned}
\Delta T &= \frac{1}{2}\mathrm{m}_p (w'_p)^2 - \frac{1}{2}\mathrm{m}_p (w_p)^2 \\
&= \frac{1}{2}\mathrm{m}_p (w_p + 2u)^2 - \frac{1}{2}\mathrm{m}_p w_p^2 \\
&= 2\mathrm{m}_p u (w_p + u) \qquad (2.98)
\end{aligned}
$$

and for Overtaking:

$$\begin{aligned}\Delta \mathrm{T} &= \frac{1}{2}\mathrm{m}_p(w_p')^2 - \frac{1}{2}\mathrm{m}_p(w_p)^2 \\ &= \frac{1}{2}\mathrm{m}_p(w_p - 2u)^2 - \frac{1}{2}\mathrm{m}_p w_p^2 \\ &= -2\mathrm{m}_p u(w_p - u). \qquad (2.99)\end{aligned}$$

We next calculate the mean rate-of-change of a particle's kinetic energy between head-on and overtaking type scatterings.

This gives

$$\frac{\Delta \mathrm{T}}{\Delta t} = \frac{1}{2}\left[\left(\frac{\Delta \mathrm{T}}{\Delta t}\right)_{\text{H-on}} + \left(\frac{\Delta \mathrm{T}}{\Delta t}\right)_{\text{overt}}\right]. \qquad (2.100)$$

The rates for the two types of collision are roughly

$$\left(\frac{1}{\Delta t}\right)_{\text{H-on}} \approx \frac{w_p + u}{\mathrm{L}} \qquad (2.101)$$

$$\left(\frac{1}{\Delta t}\right)_{\text{Overt}} \approx \frac{w_p - u}{\mathrm{L}}. \qquad (2.102)$$

Then

$$\begin{aligned}\left(\frac{\Delta \mathrm{T}}{\Delta t}\right) &= \frac{1}{2}\left[\left(\frac{w_p + u}{\mathrm{L}}\right)(2\mathrm{m}_p u(w_p + u)) + \left(\frac{w_p - u}{\mathrm{L}}\right)(-2\mathrm{m}_p u(w_p - u))\right] \\ &= \frac{1}{2}\left[\frac{2\mathrm{m}_p u}{\mathrm{L}}(w_p + u))^2 - \frac{2\mathrm{m}_p u}{\mathrm{L}}(w_p - u))^2\right] \\ &= \frac{\mathrm{m}_p u}{\mathrm{L}}(4 w_p u) \qquad (2.103)\end{aligned}$$

or

$$\dot{T} = \frac{4\mathrm{m}_p u^2 w_p}{\mathrm{L}}.$$

This then gives

$$\begin{aligned}\frac{\dot{T}}{T} = \frac{1}{T}\frac{\Delta T}{\Delta t} &= \frac{(4\mathrm{m}_p u^2 w_p/\mathrm{L})}{(1/2\mathrm{m}_p w_p^2)} \\ &= 8\left(\frac{u}{w_p}\right)^2\left(\frac{w_p}{\mathrm{L}}\right) \quad ; [\tfrac{w_p}{\mathrm{L}} \approx \text{collision rate}]. \qquad (2.104)\end{aligned}$$

The constant in the equation is ~ 2 in the 3-D case.

We see that the mean fractional change in KE per collision is of the order of $\left(\frac{u}{w_p}\right)^2$. The result is a net gain because head-on collisions are more probable than overtaking collisions. Overtaking requires a particle to catch up with an inhomogeneity.

In the extreme relativistic case we have

$$\frac{1}{\mathrm{T}}\frac{d\mathrm{T}}{dt} = \mathrm{s}\left(\frac{\mathrm{c}}{\mathrm{L}}\right)\frac{v^2}{\mathrm{c}^2}$$

with s of order unity.

Now let F(T)dt represent the number of particles in the energy interval (T, T + dT) in a region of a turbulent magnetic field.
Conservation of particles leads to

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial \mathrm{T}}\frac{d\mathrm{T}}{dt}dt = 0 \\ \frac{dF}{dt} &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \mathrm{T}}\frac{d\mathrm{T}}{dt} = 0 \\ &= \frac{\partial F}{\partial t} + \frac{\partial}{\partial \mathrm{T}}(F\frac{d\mathrm{T}}{dt}) = 0 \end{aligned} \tag{2.105}$$

since

$$\begin{aligned} \frac{\partial}{\partial \mathrm{T}}(F\frac{d\mathrm{T}}{dt}) &= \frac{\partial F}{\partial \mathrm{T}}\frac{d\mathrm{T}}{dt} + F\frac{\partial^2 \mathrm{T}}{\partial t \partial \mathrm{T}} \\ &= \frac{\partial F}{\partial \mathrm{T}}\frac{d\mathrm{T}}{dt} \quad ;\text{[to first order]}. \end{aligned} \tag{2.106}$$

Now if $\frac{\partial F}{\partial t} = 0$, we get

$$\begin{aligned} \Rightarrow \quad \frac{\partial}{\partial \mathrm{T}}(F\frac{d\mathrm{T}}{dt}) &= 0 \\ \int d(F\frac{d\mathrm{T}}{dt}) &= \mathrm{const} \quad ;[\text{with } \frac{\partial \mathrm{const}}{\partial \mathrm{T}} = 0] \\ F\frac{d\mathrm{T}}{dt} &= \mathrm{const} \end{aligned} \tag{2.107}$$

and thus

$$F(\mathrm{T}) = \frac{\mathrm{const}}{\frac{d\mathrm{T}}{dt}}. \tag{2.108}$$

Now let

$$\frac{d\mathrm{T}}{dt} = \alpha \mathrm{T} \quad (\mathrm{T}_\circ < \mathrm{T} < \mathrm{T}_1) \quad (\alpha(\mathrm{s}^{-1})$$

then

$$F(\mathrm{T}) = \frac{\mathrm{const}}{\alpha \mathrm{T}}. \tag{2.109}$$

Consider a rate of injection $\mathrm{N}_\circ$ of particles at the energy $\mathrm{T}_\circ$

$$\begin{aligned} \mathrm{N}_\circ &= F(\mathrm{T}_\circ)\left(\frac{d\mathrm{T}}{dt}\right)_\circ = F(\mathrm{T}_\circ)\alpha \mathrm{T}_\circ \\ \Rightarrow \quad F(\mathrm{T}_\circ) &= \frac{\mathrm{N}_\circ}{\alpha \mathrm{T}_\circ} \end{aligned} \tag{2.110}$$

and therefore the condition in (2.107) is

$$\text{const} = \mathrm{N}_\circ$$

$$\Rightarrow \quad F(\mathrm{T}) = \frac{\mathrm{N}_\circ}{\alpha \mathrm{T}} = \frac{\mathrm{N}_\circ}{\alpha \mathrm{T}_\circ}\frac{\mathrm{T}_\circ}{\mathrm{T}}.$$

This then gives

$$F(\mathrm{T}) \propto \mathrm{T}^{-1}. \tag{2.111}$$

In general the spectrum is not as flat as this result would suggest meaning the decline in number of particles with energy T is not as gradual as the result suggests. If a particle has a mean life time in the shock region the spectrum changes.

Suppose a particle has a mean life in such a region with characteristic time τ.

Then

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial \mathrm{T}}\frac{d\mathrm{T}}{dt}dt + \frac{F}{\tau}dt = 0 \\ \frac{dF}{dt} &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \mathrm{T}}\frac{d\mathrm{T}}{dt} + \frac{F}{\tau} = 0 \\ &= \frac{\partial F}{\partial t} + \frac{\partial}{\partial \mathrm{T}}(F\frac{d\mathrm{T}}{dt}) + \frac{F}{\tau} = 0. \end{aligned} \tag{2.112}$$

In a steady state with $\frac{\partial F}{\partial t} = 0$:

$$\begin{aligned} \frac{\partial}{\partial \mathrm{T}}\left(F\frac{d\mathrm{T}}{dt}\right) + \frac{F}{\tau} &= 0 \\ d\left(F\frac{d\mathrm{T}}{dt}\right) + \frac{F}{\tau}d\mathrm{T} &= 0 \\ \alpha \mathrm{T} dF + F d(\alpha \mathrm{T}) + \frac{F}{\tau}d\mathrm{T} &= 0 \\ \frac{dF}{F} + \frac{d\mathrm{T}}{\mathrm{T}} + \frac{1}{\alpha\tau}\frac{d\mathrm{T}}{\mathrm{T}} &= 0 \end{aligned}$$

Integration gives:

$$\begin{aligned} (\ln F - \ln F_\circ) + (\ln \mathrm{T} - \ln \mathrm{T}_\circ) + \frac{1}{\alpha\tau}(\ln \mathrm{T} - \ln \mathrm{T}_\circ) &= 0 \\ \ln(F\mathrm{T}^{(1+\frac{1}{\alpha\tau})}) &= \ln(F_\circ \mathrm{T}_\circ^{(1+\frac{1}{\alpha\tau})}) \\ F\mathrm{T}^{(1+\frac{1}{\alpha\tau})} &= F_\circ \mathrm{T}_\circ^{(1+\frac{1}{\alpha\tau})} \end{aligned}$$

and finally, resulting in

$$\begin{aligned} F &= F_\circ \left(\frac{\mathrm{T}_\circ}{\mathrm{T}}\right)^{(1+\frac{1}{\alpha\tau})} \\ &= \frac{\mathrm{N}_\circ}{\alpha \mathrm{T}_\circ}\left(\frac{\mathrm{T}_\circ}{\mathrm{T}}\right)^{(1+\frac{1}{\alpha\tau})}. \end{aligned} \tag{2.113}$$

This spectrum then is steeper than the previous one by a factor $\frac{1}{\alpha\tau}$. This means that the population of accelerated particles depends on the influx of 'fresh'particles into the shock acceleration region, the initial energies of these particles and the time they remain in the acceleration region.

2.8 Shock-waves and fast particles: Shock drift acceleration

The acceleration of charged particles as a results of criss-crossing magnetohydrodynamic wave fronts has been extensively documented, see for example Toptyghin (1980)(review article). In this section we will concentrate however, only on the reflection of a charged particle with a high velocity, oblique shock (see Figure 20), resulting in acceleration of the particle through the so-called shock drift acceleration process or fast-Fermi process (Benz 1994).

If charged particles have gyroradii which are much smaller than the thickness of a magnetohydrodynamic disturbance propagating through the plasma, the shock will act as a magnetic mirror since there is an increase in the magnetic field across a shock for fast-mode shocks (Benz 1994). The acceleration of the particle in the particle-shock interaction is basically the result of the conservation of the magnetic moment, as was discussed earlier. It can be shown that, from the condition

$$\frac{\mathrm{p}_{\perp}^2}{\mathrm{B}} = \mathrm{const}$$

for non-relativistic particles

$$\begin{aligned} \mathrm{E}_{\perp} &= \frac{1}{2}\mathrm{m}_\circ v_{\perp}^2 \quad \Rightarrow \quad 2\mathrm{E}_{\perp}\mathrm{m}_\circ = (\mathrm{m}_\circ v_{\perp})^2 \\ \mathrm{p}_{\perp}^2 &= 2\mathrm{m}_\circ\mathrm{E}_{\perp} \qquad (2.114) \\ \Rightarrow \frac{2\mathrm{m}_\circ\mathrm{E}_{\perp}}{\mathrm{B}} &= \mathrm{const} \\ \Rightarrow \frac{\mathrm{E}_{\perp 2}}{\mathrm{B}_2} &= \frac{\mathrm{E}_{\perp 1}}{\mathrm{B}_1} \\ \Rightarrow \frac{\mathrm{E}_{\perp 2}}{\mathrm{E}_{\perp 1}} &= \frac{\mathrm{B}_2}{\mathrm{B}_1}. \qquad (2.115) \end{aligned}$$

Similarly for relativistic particles from the condition :

$$\frac{\mathrm{p}_{\perp}^2}{\mathrm{B}} = \mathrm{const}$$

where

$$\mathrm{E}_{\perp} \approx \mathrm{p}_{\perp}\mathrm{c} \quad \Rightarrow \quad \mathrm{p}_{\perp} = \frac{\mathrm{E}_{\perp}}{\mathrm{c}}.$$

Then

$$\frac{(\mathrm{E}_{\perp}/\mathrm{c})^2}{\mathrm{B}} = \mathrm{const}$$

$$\Rightarrow \quad \frac{E_{\perp}^2}{B} = \text{const.} \tag{2.116}$$

This then gives

$$\begin{aligned}
\frac{E_{\perp 2}^2}{B_2} &= \frac{E_{\perp 1}^2}{B_1} \\
\Rightarrow \quad \frac{E_{\perp 2}}{E_{\perp 1}} &= \left(\frac{B_2}{B_1}\right)^{1/2}.
\end{aligned} \tag{2.117}$$

Therefore we have

$$\begin{aligned}
\text{non-relativistic:} \quad \frac{E_{\perp 2}}{E_{\perp 1}} &= \frac{B_2}{B_1} \\
\text{relativistic} \quad \frac{E_{\perp 2}}{E_{\perp 1}} &= \left(\frac{B_2}{B_1}\right)^{1/2}.
\end{aligned} \tag{2.118}$$

From these equations one can see that the interaction of the particles with magnetic compressions in shock fronts in astrophysical plasmas will result in the increase of the energy of the particles. The probability that a charged particle is reflected by a shock front is determined by estimating the critical pitch angle of the particle with respect to the field as a function of the magnetic compression in a shock front. We showed earlier that for particles trapped between approaching magnetic inhomogeneities (magnetic compressions, which can be produced by shock fronts propagating through the plasma), the particles will be trapped if

$$\left(\frac{v_{\parallel\circ}}{v_{\perp\circ}}\right)^2 < \frac{B_m}{B_\circ} - 1. \tag{2.119}$$

We have $v_{\parallel\circ}^2 = v_{p\circ}^2 - v_{\perp\circ}^2$ so

$$\begin{aligned}
\frac{v_{p\circ}^2 - v_{\perp\circ}^2}{v_{\perp\circ}^2} &< \frac{B_m}{B_\circ} - 1 \\
\left(\frac{v_{p\circ}}{v_{\perp\circ}}\right)^2 &< \frac{B_m}{B_\circ} \\
\left(\frac{v_{\perp\circ}}{v_{p\circ}}\right)^2 &> \frac{B_\circ}{B_m} \\
\sin^2\theta &> \frac{B_\circ}{B_m} \\
\cos\theta_\circ &> \left(1 - \frac{B_\circ}{B_m}\right)^{1/2}.
\end{aligned} \tag{2.120}$$

This means that if, for example, $\frac{B_\circ}{B_m} \approx 4$, then $\theta_\circ > 30°$. The trapping of a particle between moving inhomogeneities (shocks) in a plasma secures the continuous interaction of the particle with the shock, which will result in the net gain of kinetic energy of the particle.

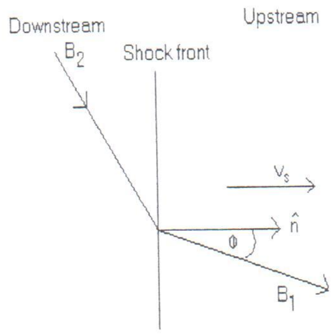

Figure 20 : An oblique shock moves at an angle with respect to the upstream B-field.

Energy may be gained by particles that are reflected repeatedly by the shock while conserving their magnetic moments $p_{\perp}^{2}/B$. A detailed analysis showed that the maximum energy transfer to a particle in a single crossing of the shock is (Toptyghin 1980, Priest & Forbes 2000)

$$\frac{\Delta E_{max}}{E_i} = \left(4\left(\frac{B_d}{B_u}\right) - 3\right) \text{ (thermal)}$$

and

$$\frac{\Delta E_{max}}{E_i} = \left(2\left(\frac{B_d}{B_u}\right) - 1\right) \text{ (relativistic)}$$

where B_d and B_u are the up and down stream magnetic fields respectively. Strong shocks give $B_d/B_u \rightarrow 4$, and thus the energy of thermal and mildly relativistic electrons can be increased by factors of $(\Delta\ E_{max}/E_i) \rightarrow 13$ and $(\Delta\ E_{max}/E_i) \rightarrow 7$ respectively, in a single encounter with a strong shock.

In this chapter a description was given of a variety of scenarios in which particle acceleration can occur in highly conducting and turbulent media. The mhd propeller in AE Aquarii, expelling blobs of plasma from the fast rotating white dwarf magnetosphere, is conducive to mechanisms such as reconnection, adiabatic compression as well as the generation of shock waves in the plasma. The efficiency of these processes in producing populations of high energy particles is limited, but as will be shown the model for the radio emission from AE Aqr does not require high efficiency. Even limited efficiency in terms of acceleration of some fraction of the thermal population of electrons in the plasma blobs is sufficient to drive the non-thermal outbursts from AE Aquarii, from the radio to infrared frequencies. These aspects will be addressed in the next chapter. The discussions that follow form the foundation of the model developed to explain the synchrotron flares from AE Aquarii, as is documented in the paper, *Meintjes P.J. & Venter L.A. 2003, Modelling the continuous radio outbursts in AE Aquarii, MNRAS, 341, 891.* The paper is included at the back.