Energy Density and Luminosity

We showed that

$$u(\Omega) = \frac{I}{c}$$

$$u = \int_{\Omega} u(\Omega)\, d\Omega$$

There is a relation between the energy density and the luminosity emitted by the source. The relation depends whether the observer is positioned

- outside the source
- at a large distance with respect to the size of the source.
- inside the source

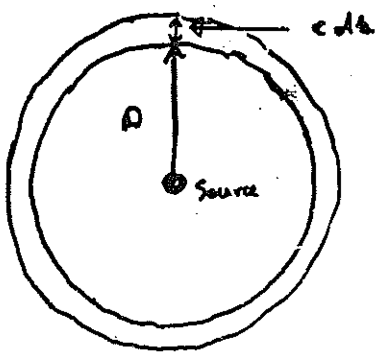
Outside the Source

Assume to be at a distance D from the source. At this position consider a shell of surface $A = 4\pi D^2$ and a thickness $dr = c\,dt$

In a time dt the source has emitted energy

$$dE = L\, dt$$

Due to energy conservation the same energy flows through the shell in a time dt.



$$\boxed{dV_{shell} = 4\pi D^2 c\, dt}$$

Therefore the energy density is the total energy content within the shell

$$u = \frac{E}{V}$$

$$= \frac{L\,dt}{4\pi D^2 c\,dt}$$

$$= \frac{L}{c \times 4\pi D^2}$$

$$= \frac{1}{c}\,\frac{L}{4\pi D^2}$$

$$= \frac{1}{c}\,F \qquad F = \text{flux}$$

Notice that

$$u = u(\Omega)\,\Delta\Omega = \frac{1}{c}\,\frac{L}{4\pi D^2}$$

$$\frac{I}{c}\,\Delta\Omega = \frac{1}{c}\,\frac{L}{4\pi D^2}$$

$$\Rightarrow \quad I\,\Delta\Omega = \frac{L}{4\pi D^2} = F$$

Hence the measured energy density (u_{meas}) is :

$$u_{meas} = \frac{1}{c}\,\frac{L}{4\pi D^2} \qquad \text{flux}$$

$$= \frac{1}{c}\left\{I\,\Delta\Omega\right\}$$

$$= \frac{I}{c}\,\Delta\Omega$$

$$= u(\Omega)\,\Delta\Omega$$

$$= u_{tot} \times \frac{\Delta\Omega}{4\pi} \qquad u(\Omega) = \frac{u_{tot}}{4\pi}$$

Radiation trapped inside a uniformly emitting source

Consider a large source that is optically thin and homogeneous. In this case the energy density depends on the mean escape time of the photons from the source.

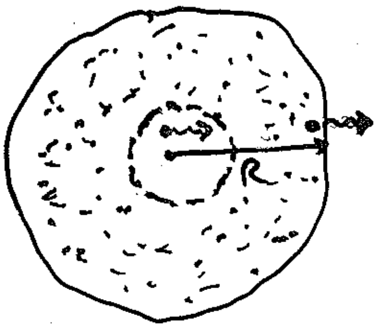
For a given fixed Luminosity the larger the escape time the larger the energy density, i.e. energy gets trapped inside the source. Therefore

$$u = \frac{L\langle t_{esc}\rangle}{V}$$

For a source of given size the escape time of a photons born at the centre is

$$t_{esc} \approx \frac{R}{c} \qquad (\tau < 1)$$

Consider photons born near the edge of a source of radius R. This photon can be directed outward or inward. The probability to be directed outward is $P \sim \frac{1}{2}$ and hence to escape immediately the probability is $\sim \frac{1}{2}$.



The probability of the photon traveling the full radius of the source without scattering is less than $\frac{1}{2}$. Therefore one can see that the escape time

$$\boxed{t_{esc} < \frac{R}{c}}$$

$$P = e^{-\tau} = e^{-n\sigma L} = e^{-\frac{R}{l}} \quad L \to R$$

$$\approx 0.36 \quad l \to R$$

* Photon created at centre will be absorbed. Only photons born in "red zone" will escape within t_{esc}

The average escape time is approximately

$$\langle t_{esc} \rangle \sim \frac{3}{4} \frac{R}{c}$$

Let's try to explain this :

For an optically thin source $(\tau < 1)$ the number of scatterings and optical depth are related

$$\tau = N = n \sigma L$$

$$= \frac{N}{\frac{4\pi}{3} R^3} \sigma L$$

$$= \frac{3}{4} \left(\frac{N \sigma}{\pi R^2} \right) \frac{L}{R}$$

$$= \frac{3}{4} \left(\frac{N \sigma}{\pi R^2} \right) \qquad L \to R$$

$$\hookrightarrow \; \leq 1$$

But for sources with $\tau < 1$ and $\tau > 1$ respectively

$$\langle t_{esc} \rangle = N \frac{R}{c} = \tau \frac{R}{c} \qquad \tau < 1$$

$$\langle t_{esc} \rangle = N \frac{R}{c} = \tau^2 \frac{R}{c} \qquad \tau > 1$$

Therefore, for $\tau < 1$

$$\langle t_{esc} \rangle = \tau \frac{R}{c}$$
$$\approx \frac{3}{4} \frac{R}{c}$$

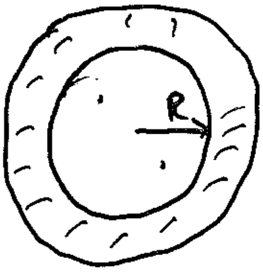
Therefore

$$u = \frac{L \langle t_{esc} \rangle}{V}$$
$$= \frac{L}{V} \times \frac{3}{4} \frac{R}{c}$$
$$= \frac{L}{\frac{4\pi}{3} R^3} \times \frac{3}{4} \frac{R}{c}$$
$$= \frac{3L}{4\pi R^3} \times \frac{3}{4} \frac{R}{c}$$
$$= \frac{9}{4} \frac{L}{4\pi R^2 c}$$

One can see that in this case the energy density is a factor 9/4 higher. This is because radiation gets trapped inside as a result of scattering.

Inside a uniformly emitting shell.

Assume that we are surrounded by a uniformly emitting shell of radius R. The shell emits a luminosity L (isotropically). The amount of energy it produces in a time dt is



$$E \approx L\, dt$$

The energy density within the shell is

$$U \approx \frac{E_{shell}}{V}$$

$$= \frac{L_{shell} \times dt}{4\pi R^2 c\, dt}$$

$$= \frac{L_{shell}}{4\pi R^2 c}$$

The luminosity within the shell $r < R$ will be everywhere the same based on geometry.

The relation between Flux and Intensity

The flux emitted by an object subtending a solid angle Ω is

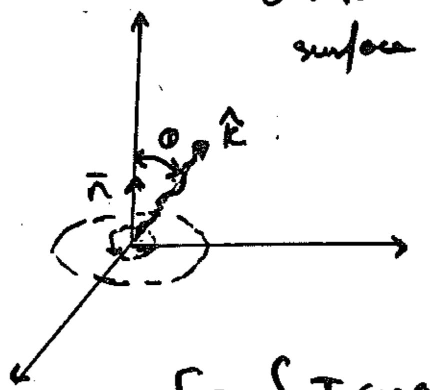
$$F = \int_\Omega I \cos\theta \, d\Omega$$

One can show that the flux from the surface of a star for an observer far away is

$$F = \pi I$$

Proof:

For an observer far away the star appears like a disc. Photons with wave vector $\hat{k}$ between $\theta = 0$ and $\theta = 90°$ can cut through the surface. Hence



$$0 \leq \phi \leq 2\pi$$
$$0 \leq \theta \leq \frac{\pi}{2}$$

$$F = \int_\Omega I \cos\theta \, d\Omega \qquad d\Omega = \sin\theta \, d\theta \, d\phi$$

$$= I \int_\Omega \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= I \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= 2\pi I \times \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$

$$= 2\pi I \times \frac{1}{2}\left[\sin^2 \pi/2 - \sin^2 0\right]$$

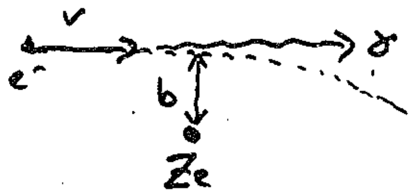
$$= \pi I$$

Bremsstrahlung and Black Body

Bremsstrahlung

Here we follow an approximate derivation. For simplicity we will consider an electron-proton plasma. The following definitions will be clarified

b : impact parameter
v : velocity of electron
n_e : electron number density
n_p : proton number density
T : Temperature of plasma



$$(m v^2) \sim kT$$

$$v \sim \sqrt{\frac{kT}{m}}$$

We will calculate the total power as well as the spectrum of bremsstrahlung emission. The procedure will be divided into a few steps.

1.) We consider the interaction between an electron and proton only when the electron passes close to a proton. The interaction time is

$$\tau \sim \frac{b}{v}$$

2.) During the interaction we consider the acceleration to be constant and equal to

$$m_e a \approx F_e$$
$$\approx \frac{e^2}{b^2}$$

Then

$$a \approx \frac{e^2}{m_e b^2}$$

3.) From the Larmor formula we get

$$\begin{aligned} P &= \frac{2e^2a^2}{3c^3} \\ &= \frac{2e^2}{3c^3}\left(\frac{e^2}{m_e b^2}\right)^2 \\ &= \frac{2e^2}{3c^3}\frac{e^4}{m_e^2 b^4} \\ &= \frac{2}{3}\frac{e^6}{m_e^2 c^3 b^4} \\ &\approx \frac{e^6}{m_e^2 c^3 b^4} \end{aligned}$$

4.) Since there is a characteristic time τ for the interaction there is also a characteristic frequency

$$\begin{aligned} \omega &\sim \frac{1}{\tau} \\ &= \frac{v}{b} \end{aligned}$$

5.) Therefore

$$\begin{aligned} P(\omega) &= \frac{P}{\omega} \\ &= \frac{\left[\frac{e^6}{m_e^2 c^3 b^4}\right]}{\omega} \\ &= \left[\frac{e^6}{m_e^2 c^3 b^4}\right] * \left[\frac{b}{v}\right] \\ &\approx \frac{e^6}{m_e^2 c^3 b^3 v} \end{aligned}$$

$\frac{1}{\omega}$

6.) We can estimate the impact factor b from the density of protons (targets)

$$b \sim (n_p)^{-1/3}$$

$$b^3 \sim n_p^{-1}$$

$$\Rightarrow b^3 n_p \approx 1 \qquad \Rightarrow n_p \sim \frac{1}{b^3}$$

7.) The emissivity $j(\omega)$ will be the power emitted by a single e^- multiplied by the number density of electrons. If the emission is isotropic we will have to multiply by $\frac{1}{4\pi}$ since the emissivity is directional

$$j(\nu) \to \text{erg cm}^{-3}\,\text{s}^{-1}\,\text{Hz}^{-1}\,\text{sr}^{-1}$$

Then

$$j(\omega) = \frac{n_e}{4\pi} P(\omega)$$

$$= \frac{n_e}{4\pi} \frac{e^6}{m_e^2 c^3 b^3 v}$$

$$= \frac{n_e}{4\pi} \frac{1}{b^3} \frac{e^6}{m_e^2 c^3} \frac{1}{v}$$

$$= \frac{n_e}{4\pi} n_p \frac{e^6}{m_e^2 c^3} \left(\frac{kT}{m_e}\right)^{-1/2}$$

$$\approx \frac{n_e n_p}{4\pi} \frac{e^6}{m_e^2 c^3} \left(\frac{m_e}{kT}\right)^{1/2}$$

8.) The total emissivity is

$$j = \int_0^{\omega_{max}} j(\omega)\, d\omega$$

{ ω_{max} the frequency where maximum emission is emitted }

Notice the integral depends on ω_{max}. What is ω_{max}?

One possibility is to set

$$\hbar \omega_{max} \sim kT \Rightarrow \omega_{max} \sim \left(\frac{kT}{\hbar}\right)$$

This means that an electron cannot emit a photon with energy larger than $\hbar\omega_{max}$, i.e limited by the average energy kT. Hence by doing this we ignore electrons with energies greater than kT

$$j = \int_0^{\omega_{max}} j(\omega)\, d\omega$$

$$\approx j(\omega)\, \omega_{max}$$

$$\approx \frac{n_e n_p}{4\pi} \frac{e^6}{m_e^2 c^3} \left(\frac{m_e}{kT}\right)^{1/2} \left(\frac{kT}{\hbar}\right)$$

$$\approx \frac{n_e n_p}{4\pi} \frac{e^6}{m_e^2 c^3} \frac{m_e^{1/2}}{\hbar} (kT)^{1/2}$$

$$\approx \frac{n_e n_p}{4\pi} \frac{e^6}{m_e^2 c^3}, \frac{(m_e kT)^{1/2}}{\hbar}$$

The exact result must contain contributions from electrons with energy $\epsilon > kT$, e.g from the tail of the M B distribution. Before we continue we give a brief summary of the M-B and other distributions.

Distribution Functions : Thermal & Non-Thermal

Thermal Plasmas

The number of particles per state for a thermal plasma or gas is given by the M-B distribution function. This distribution is the density of particles in phase space

$$\text{Phase space volume element} = d^3r\, d^3p$$
$$= dx\,dy\,dz\; dp_x\, dp_y\, dp_z$$

$$dV = d^3r\, d^3p$$

$$dN = f(\bar{p},\bar{r})\, d^3r\, d^3p$$

$$\underbrace{f(\bar{p},\bar{r})}_{\text{Density function in phase space}} \quad [\, m^{-3}\,(kg\, m\, s^{-1})^{-3} \,]$$

We can also construct a velocity distribution

$$f(\bar{v},\bar{r}) = \underbrace{m_p^3}_{\text{mass of particle}}\, f(\bar{p},\bar{r})$$

For most scenario's the distribution function is not dependent on position.

$$f(\bar{p},\bar{r}) \rightarrow f(\bar{p})$$

Then

$$N = \int d^3r \int f(\bar{p})\, d^3p$$
$$= V \int f(\bar{p})\, d^3p$$
$$\Rightarrow \quad n = \frac{N}{V} = \int f(\bar{p})\, d^3p$$
$$dn = f(\bar{p})\, d^3p$$

One can show that

$$f(\bar{p}) = n \left(\frac{1}{2\pi m k T}\right)^{3/2} e^{-\frac{p^2}{2mkT}} \quad [m^{-3}(kg\,m\,s^{-1})^{-3}]$$

$$f(\bar{v}) = m^3 f(\bar{p})$$
$$= n \left(\frac{m}{2\pi k T}\right)^{3/2} e^{-\frac{p^2}{2mkT}} \quad [m^{-3}(m.s^{-1})^{-3}]$$

Notice that

$$f(\bar{p}) = \frac{dn}{d^3p} \qquad \text{and} \qquad f(\bar{v}) = \frac{dn}{d^3v}$$

Then one can write

$$\frac{dn}{d^3p} = \frac{dn}{4\pi p^2 dp} = f(\bar{p})$$
$$\frac{\partial n}{\partial p} = 4\pi p^2 f(\bar{p})$$

In similar fashion

$$\frac{\partial n}{d^3v} = \frac{\partial n}{4\pi v^2 dv} = f(\bar{v})$$
$$\frac{dn}{dv} = 4\pi v^2 f(\bar{v})$$

Hence

$$\frac{dn}{dp}\ (m^{-3}\,(\text{mom})^{-1}) = 4\pi p^2 f(\bar{p})$$

$$\frac{dn}{dv}\ (m^{-3}\,(\text{velocity})^{-1}) = 4\pi v^2 f(\bar{v})$$

Therefore the density of particles with momenta and velocity within dp (or dv) are

$$dn(p) = 4\pi p^2 f(\bar{p})\,dp \qquad m^{-3} \text{ within } (p, p+dp)$$

$$dn(v) = 4\pi v^2 f(\bar{v})\,dv \qquad m^{-3} \text{ within } (v, v+dv)$$

One can extend this to probability

$$dP(p) = \frac{dn(p)}{n} = 4\pi p^2 \left(\frac{1}{2\pi mkT}\right)^{3/2} e^{-\frac{p^2}{2mkT}}\,dp$$

$$dP(v) = \frac{dn(v)}{n} = 4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{p^2}{2mkT}}\,dv$$

Non - Thermal Plasmas

In most astrophysical environments particles can be accelerated to energies exceeding the average thermal energy. Therefore in these environments a fraction of the total particle population may have a different distribution, in most cases a power law spectrum.

$$N(\gamma) = K \gamma^{-p} \qquad \gamma = \left(\frac{E}{mc^2}\right)$$

Here $N(\gamma)$ is usually the density of particles per unit energy within the range $[\gamma, \gamma + d\gamma]$

$$N(\gamma) = \frac{dn}{d\gamma} = K \gamma^{-p} \quad [m^{-3} \gamma^{-1}]$$

Therefore the units of the constant are

$$K \quad [m^{-3} \gamma^{p-1}]$$

The form of the power-law above is handy, just remember it comes from

$$N(E) = \underset{\uparrow}{\text{Const}} \; E^{-p} \qquad [m^{-3} \text{ Joule}^{-1}]$$

$$\text{Const} \Rightarrow [m^{-3} \text{ Joule}^{p-1}]$$

Relativistic Maxwellian

We showed earlier

$$\frac{\partial n}{\partial p} = 4\pi p^2 \, f(\bar{p})$$

For particles that are relativistic

$$\varepsilon = \sqrt{p^2c^2 + m_0^2c^4}$$

$$\varepsilon = \gamma mc^2 \approx \sqrt{p^2c^2 + m^2c^4}$$

$$\varepsilon \approx pc \qquad pc \gg m^2c^4$$

$$c\,\frac{\partial n}{\partial (pc)} \sim 4\pi \frac{(pc)^2}{c^2} \,\mathrm{Const}\, e^{-\frac{\varepsilon}{kT}}$$

$$\frac{\partial n}{\partial \varepsilon} \sim 4\pi \,\mathrm{Const}\, \frac{\varepsilon^2}{c^3} \cdot e^{-\frac{\gamma mc^2}{kT}}$$

$$\sim K \varepsilon^2 e^{-\frac{\gamma}{kT/mc^2}}$$

$$\sim K \varepsilon^2 e^{-\frac{\gamma}{\Theta}} \qquad \Theta = \frac{kT}{mc^2}$$

$$N(\varepsilon) \propto \frac{\partial n}{\partial \varepsilon} \sim K \varepsilon^2 e^{-\frac{\gamma}{\Theta}}$$

$$\sim K \gamma^2 e^{-\frac{\gamma}{\Theta}} \qquad \gamma = \frac{pc}{m_pc^2}$$

The exact result is given by Rybicki & Lightman which is

$$\frac{dG}{dV\,dt\,d\omega} = \frac{16\pi e^6 Z^2}{3\sqrt{3}\, c^3 m_e^2 v}\, n_e n_i\, g_{ff}(v,\omega) \; [\mathrm{erg\, cm^{-3}\, s^{-1}\, Hz^{-1}}]$$

To take into account the whole thermal distribution of electrons the probability of electrons having speeds within the interval $[v, v+dv]$ is

$$dP(v) = \frac{dn(v)}{n}$$

$$= \mathrm{Const}\; v^2\, e^{-\frac{mv^2}{2kT}}$$

The emissivity can be obtained by integrating over the M-B distribution. This integration depends on the integration limits. The limits are determined as follows:

The photon emitted by the deflected electron cannot exceed its own KE

$$h\nu < \tfrac{1}{2} m_e v^2$$

$$v^2 > \frac{2h\nu}{m_e}$$

$$v > \left(\frac{2h\nu}{m_e}\right)^{1/2}$$

$$v_{min} \approx \left(\frac{2h\nu}{m_e}\right)^{1/2}$$

The integration then becomes:

$$\left\langle \frac{dG}{dV\,dt\,d\omega} \right\rangle = \frac{\displaystyle\int_{v_{min}}^{\infty} \underbrace{j_\omega(v)}_{\left[\frac{dG}{dV\,dt\,d\omega}\right]} \, dP(v)}{\displaystyle\int_0^{\infty} dP(v)}$$

* Averaging over the M-B distribution

$$\langle A(v) \rangle = \frac{\int A(v)\, \underbrace{f(\vec{v})\, d^3v}_{dn(v)}}{\int f(\vec{v})\, d^3v}$$

$$= \frac{\frac{1}{n} \int A(v)\, dn(v)}{\frac{1}{n} \int dn(v)}$$

$$= \frac{\int A(v)\, dP(v)}{\int dP(v)}$$

The integration limits for the numerator are $v_{min} \to \infty$ while in the denominator we integrate over the whole thermal distribution. Therefore

$$\left\langle \frac{dG}{dV\,dt\,d\omega} \right\rangle = \frac{\frac{16\pi e^6}{3\sqrt{3}\, c^3 m_e^2}\, n_e n_i\, Z^2 g_{ff} \displaystyle\int_{v_{min}}^{\infty} \frac{1}{v} \times e^{-\frac{mv^2}{2kT}} \underbrace{d^3v}_{\text{isotropic}}}{\displaystyle\int_0^{\infty} e^{-\frac{mv^2}{2kT}}\, d^3v}$$

$$= \frac{\frac{16\pi e^6}{3\sqrt{3}\, c^3 m_e^2}\, n_e n_i\, Z^2 g_{ff} \displaystyle\int_{v_{min}}^{\infty} v\, e^{-\frac{mv^2}{2kT}}\, dv}{\displaystyle\int_0^{\infty} v^2 e^{-\frac{mv^2}{2kT}}\, dv}$$

The integration can now be performed:

Denominator:
$$\int_0^\infty v^2 e^{-\frac{mv^2}{2kT}}\, dv = \left[\frac{\sqrt{\pi}}{4\left(\frac{m}{2kT}\right)^{3/2}} \right]$$

Numerator: Partial integration

$$\frac{d}{dv}\left(e^{-\frac{mv^2}{2kT}}\right) = -\frac{2mv}{2kT}\, e^{-\frac{mv^2}{2kT}}$$
$$= -\frac{mv}{kT}\, e^{-\frac{mv^2}{2kT}}$$

$$\Rightarrow \quad -\frac{kT}{m}\, d\left(e^{-\frac{mv^2}{2kT}}\right) = v\, e^{-\frac{mv^2}{2kT}}\, dv$$

Therefore with this transformation
$$\int_{v_{min}}^\infty v\, e^{-\frac{mv^2}{2kT}}\, dv = -\frac{kT}{m} \int_{v_{min}}^\infty d\left(e^{-\frac{mv^2}{2kT}}\right)$$
$$= -\frac{kT}{m} \left[e^{-\frac{mv^2}{2kT}} \right]_{v_{min}}^\infty$$
$$= -\frac{kT}{m} \left[e^{-\infty} - e^{-\frac{mv_{min}^2}{2kT}} \right]$$
$$= \frac{kT}{m}\, e^{-\frac{mv_{min}^2}{2kT}}$$

Therefore the integral is

The average now is :

$$\left\langle \frac{dG}{dV\,dt\,d\omega} \right\rangle = \frac{32\sqrt{\pi}\, e^6 n_e n_i Z^2 g_{ff} \left(\frac{m}{2kT}\right)^{3/2}}{3\sqrt{3}\, c^3 m_e^2} \left(\frac{2kT}{m}\right) e^{-\frac{m v_{min}^2}{2kT}}$$

$$= \frac{32\sqrt{\pi}\, e^6 n_e n_i Z^2 g_{ff} \left(\frac{m}{2kT}\right)^{3/2} \left(\frac{m}{2kT}\right)^{-2/2} e^{-\frac{m v_{min}^2}{2kT}}}{3\sqrt{3}\, m_e^2 c^3}$$

Using $\quad v_{min}^2 = \frac{2h\nu}{m_e}$

$$\left\langle \frac{dG}{dV\,dt\,d\omega} \right\rangle = \frac{32\sqrt{\pi}\, n_e n_i e^6 Z^2 g_{ff} \left(\frac{m}{2kT}\right)^{1/2} e^{-\frac{h\nu}{kT}}}{3\sqrt{3}\, m_e^2 c^3}$$

$$= \frac{32\sqrt{\pi}\, n_e n_i e^6 Z^2 g_{ff} \left(\frac{m}{6kT}\right)^{1/2} e^{-\frac{h\nu}{kT}}}{3\, m_e^2 c^3}$$

We can now express this in terms of frequency

$$j_\omega = \frac{dj}{d\omega}$$

$$\omega = 2\pi\nu$$
$$d\omega = 2\pi\, d\nu$$

$$j_\omega = \frac{dj}{d\omega} = \frac{dj}{2\pi\, d\nu}$$

$$\Rightarrow \quad j_\nu = \frac{dj}{d\nu} = 2\pi\, j_\omega$$

$$j_\nu = \left\langle \frac{dG}{dV\,dt\,d\nu} \right\rangle$$
$$j_\omega = \left\langle \frac{dG}{dV\,dt\,d\omega} \right\rangle$$

Hence

$$j_\nu = \left\langle \frac{dG}{dVol\,dt\,d\nu} \right\rangle = 2\pi \left\langle \frac{dG}{dVol\,dt\,d\omega} \right\rangle$$

$$j_\nu = \frac{2\pi \times 8\sqrt{\pi}\, n_e n_i Z^2 e^6 g_{ff} \left(\frac{m}{6kT}\right)^{1/2}}{3 \times m_e^2 c^3} e^{-\frac{h\nu}{kT}}$$

$$= \frac{32\pi}{3} \frac{n_e n_i Z^2 e^6 g_{ff}}{m_e^2 c^3} \left(\frac{4\pi m}{6kT}\right)^{1/2} e^{-\frac{h\nu}{kT}}$$

$$= \frac{32\pi}{3} \frac{n_e n_i Z^2 e^6 g_{ff}}{m_e^2 c^3} \left(\frac{2\pi m}{3kT}\right)^{1/2} e^{-\frac{h\nu}{kT}}$$

$[erg\ cm^{-3}\ s^{-1}\ Hz^{-1}]$

This eqn assumes isotropic emission over 4π steradians. For beamed emission

$$j_\nu(\Omega) = \frac{j_\nu}{4\pi}$$

$$= \frac{1}{4\pi} \times \left[\frac{32\pi}{3} \frac{n_e n_i e^6 Z^2 g_{ff}}{m_e^2 c^3} \left(\frac{2\pi m}{3kT}\right)^{1/2} e^{-\frac{h\nu}{kT}} \right]$$

$$= \frac{8}{3} \times \left(\frac{2\pi}{3}\right)^{1/2} \frac{n_e n_i e^6 Z^2 g_{ff}}{m_e^2 c^3} \left(\frac{m}{kT}\right)^{1/2} e^{-\frac{h\nu}{kT}}$$

$$= 5.4 \times 10^{-39} Z^2 n_e n_i T^{-1/2} g_{ff} e^{-\frac{h\nu}{kT}} \quad [erg\ cm^{-3}\ s^{-1}\ Hz^{-1}\ sr^{-1}]$$

The total emission coefficient is

$$j(\Omega) = \int j_\nu(\Omega)\, d\nu$$

$$= \frac{8}{3} \left(\frac{2\pi}{3}\right)^{1/2} \frac{n_e n_i e^6 Z^2 g_{ff}}{m_e^2 c^3} \left(\frac{m}{kT}\right)^{1/2} \int_0^\infty e^{-\frac{h\nu}{kT}} d\nu$$

$$= \frac{8}{3} \left(\frac{2\pi}{3}\right)^{1/2} \frac{n_e n_i e^6 Z^2 g_{ff}}{m_e^2 c^3} \left(\frac{m}{kT}\right)^{1/2} \left[\frac{e^{-\frac{h\nu}{kT}}}{-\frac{h}{kT}} \right]_0^\infty$$

This gives

$$j(\mathcal{A}) = \frac{8}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{n_e n_i e^6 Z^2 g_{ff}}{m_e^2 c^3} \left(\frac{m}{kT}\right)^{1/2} \left(\frac{kT}{\hbar}\right) \cdot \left[e^{-\infty} - e^{-0}\right]$$

$$= \frac{8}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{n_e n_i e^6 Z^2 g_{ff}}{m_e^2 c^3} \frac{(mkT)^{1/2}}{\hbar}$$

$$= \frac{4}{3\pi}\left(\frac{2\pi}{3}\right)^{1/2} \frac{n_e n_i e^6 Z^2 g_{ff}}{m_e^2 c^3} \frac{(mkT)^{1/2}}{h} \quad [\mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}}]$$

$\hbar = \frac{h}{2\pi}$

$$\approx 1.13\times10^{-28}\, n_e n_i Z^2 g_{ff} T^{1/2} \quad \mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}}$$

Summary:

$$j_\nu(\mathcal{A}) = 5.4\times10^{-39} Z^2 n_e n_i T^{-1/2} g_{ff}\, e^{-\frac{h\nu}{kT}} \quad [\mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}\ Hz^{-1}}]$$

$$j(\mathcal{A}) = 1.13\times10^{-28} Z^2 n_e n_i T^{1/2} g_{ff} \quad [\mathrm{erg\ cm^{-3}\ s^{-1}\ sr^{-1}}]$$

Free - Free Absorbtion

If the underlying particle distribution is a Maxwellian we can use Kirchoff's law to determine the absorbtion coefficient. The brightness of a BB emitter is

$$S_\nu = B_\nu = \frac{j_\nu}{\alpha_\nu}$$
$$= \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

Now it is a simple matter to determine α_ν if we know j_ν. This can only be done if the particle distribution is a Maxwellian

$$\alpha_\nu^{ff}(\text{cm}^{-1}) = \frac{j_\nu}{B_\nu} \frac{[\text{erg cm}^{-3}\,\text{s}^{-1}\,\text{sr}^{-1}\,\text{Hz}^{-1}]}{[\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{sr}^{-1}\,\text{Hz}^{-1}]}$$

$$= \frac{\left[\frac{\frac{8}{3}\left(\frac{2\pi}{3}\right)^{1/2} n_e n_i e^6 Z^2 g_{ff} \left(\frac{m}{kT}\right)^{1/2} e^{-\frac{h\nu}{kT}}}{m_e^2 c^3}\right]}{\left[\frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}\right]}$$

$$= \frac{4}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{n_e n_i e^6 Z^2 g_{ff}}{h m_e^2 c^2} \left(\frac{m_e c^2}{kT}\right)^{1/2} \frac{\left(1 - e^{-\frac{h\nu}{kT}}\right)}{\nu^3}$$

$$= 3.7\times10^8 \frac{Z^2 n_e n_i g_{ff}}{T^{1/2}} \frac{\left(1 - e^{-\frac{h\nu}{kT}}\right)}{\nu^3}$$

When $h\nu \ll kT$ (Rayleigh - Jeans regime) this simplifies to

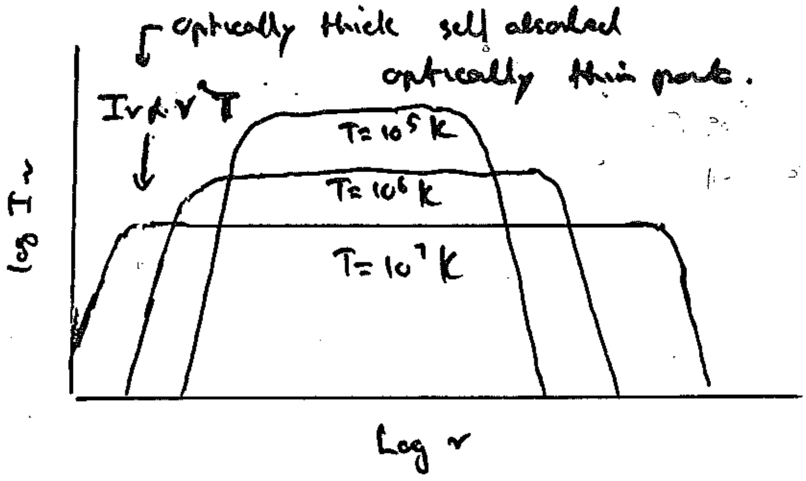
$$\alpha_\nu^{ff}(\text{cm}^{-1}) = 3.7\times10^8 \frac{Z^2 n_e n_i g_{ff}}{T^{1/2}} \left(\frac{1}{\nu^3}\right)\left(1 - \left(1 - \frac{h\nu}{kT} \cdots\right)\right)$$

$$= 3.7\times10^8 \frac{Z^2 n_e n_i g_{ff}}{T^{1/2}} \frac{1}{\nu^3}\left(\frac{h\nu}{kT}\right)$$

This reduces to

$$\alpha_\nu^{ff} = 0.018 \frac{Z^2 n_e n_i g_{ff}}{T^{3/2} \nu^2}$$

Fig 2.1 of Ghisellini shows how the bremsstrahlung intensity of a source changes with temperature



Note that

$$j_\nu \propto T^{-1/2} e^{-\frac{h\nu}{kT}} \quad \text{[optically Thin part]}$$
$$I_\nu \propto T^{-1/2} e^{-\frac{h\nu}{kT}}$$

$$I_\nu \propto \nu^2 T \quad \text{[optically thick part]}$$
↑ Rayleigh Jeans

The total integrated emission before exponential cut-off

$$j \propto T^{1/2} \quad \text{(showed earlier)}$$

Therefore, for higher temp the spectrum extends to high frequencies before it drops-off.

For the self-absorbed part $h\nu \ll kT$ or $\underbrace{\frac{h\nu}{kT} \to 0}$

$e^{-\frac{h\nu}{kT}} \to 1$

$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

$$\propto \frac{(T^{-1/2})}{\left(\frac{1}{T^{3/2}\nu^2}\right)}$$

$$\propto T^{-1/2}\, T^{3/2}\, \nu^2$$

$$\propto T\, \nu^2$$

[Rayleigh-Jeans law]

From Bremsstrahlung to Blackbody

Like any radiation process the bremsstrahlung emission has a self-absorbed part. This corresponds to optical depths $\tau_\nu \gg 1$. From the equation for the absorption coefficient

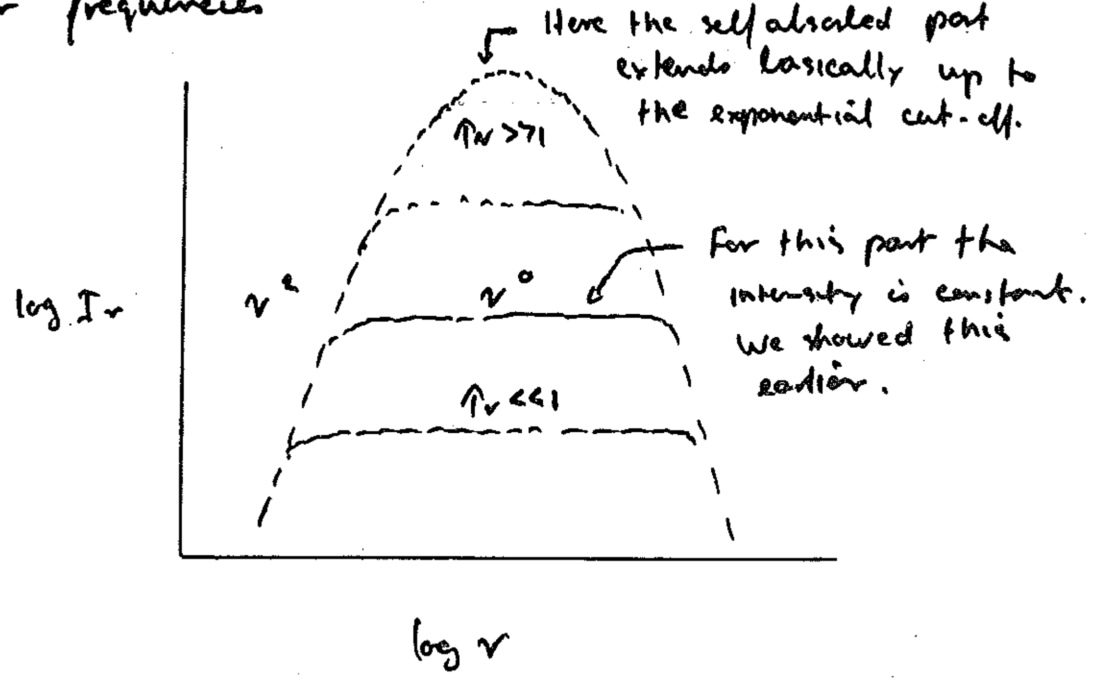
$$\alpha_\nu^{ff} \propto \frac{n_e n_i}{T^{1/2}} \frac{(1 - e^{-\frac{h\nu}{kT}})}{\nu^3}$$

or

$$\alpha_\nu^{ff} \propto \left(\frac{1}{T^{3/2}\nu^2}\right) \qquad h\nu \ll kT$$

The $\alpha_\nu^{ff} \propto \frac{1}{\nu^3}$ dependence show that the absorption will be severe at small frequencies.

By increasing the density of the emitting and absorbing photons the spectrum is self-absorbed up to larger frequencies



When all the spectrum is self-absorbed ($\tau_\nu \gg 1$ for all ν) and all the particles belong to the M-B distribution then we have a BB. One can see that the bremsstrahlung intensity becomes more and more self-absorbed until it becomes a BB. At this point increasing density does not increase the intensity any longer. This is because we only receive radiation from a thin outer shell with optical dept $\tau_\nu \to 1$. The thickness of this layer decreases as we increase the density but the emissivity increases. This is explained as follows:

$$
\begin{aligned}
I_\nu(\tau_\nu) &= S_\nu (1 - e^{-\tau_\nu}) \\
&= \frac{j_\nu}{\alpha_\nu} (1 - e^{-\tau_\nu}) \\
&= \frac{j_\nu R}{\alpha_\nu R} (1 - e^{-\tau_\nu}) \\
&= \frac{j_\nu R}{\tau_\nu} \qquad (\tau_\nu \gg 1) \\
&= j_\nu \frac{R}{\tau_\nu} \qquad \Delta R = \frac{R}{\tau_\nu}
\end{aligned}
$$

From the bremsstrahlung volume emissivity

① $I_\nu \propto j_\nu R \propto n_e n_i R$

② $\tau_\nu = \alpha_\nu R \propto n_e n_i R$

$\Rightarrow \quad I_\nu(\tau_\nu) = \frac{j_\nu R}{\tau_\nu} \propto \frac{n_e n_i R}{n_e n_i R} = \text{const} \quad (\tau_\nu \gg 1)$

Blackbody

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

Exercise: Show that the equivalent form in terms of wavelength is. Derive this from $B_\nu(T)$.

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

The two limits:

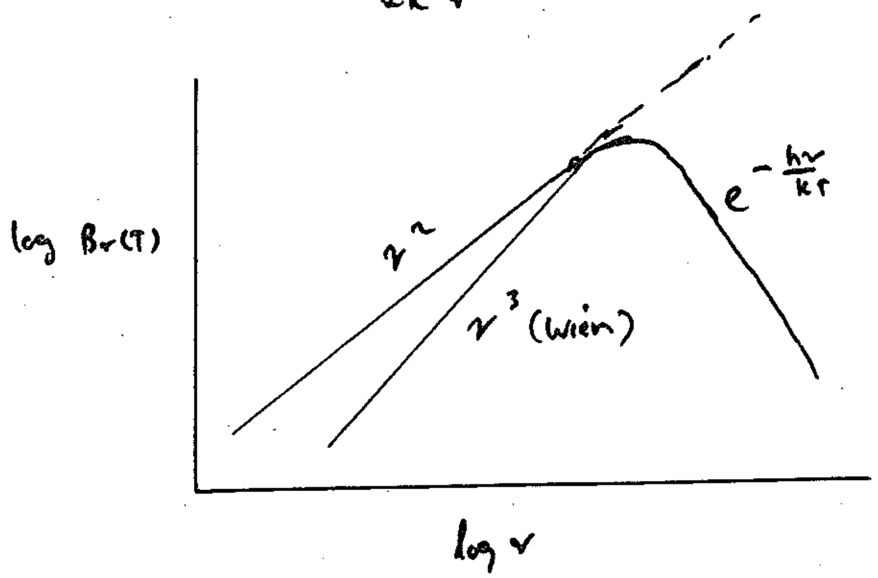
① $$I_\nu^{RJ} = \frac{2\nu^2}{c^2} kT \qquad h\nu \ll kT$$

② $$I_\nu^{W} = \frac{2h}{c^2} \nu^3 e^{-\frac{h\nu}{kT}} \qquad h\nu \gg kT$$

The brightness temperature is defined using the R-J law

$$I_\nu^{RJ} = \frac{2h\nu^2}{c^2} kT$$

$$T_b = \frac{c^2 I_\nu^{RJ}}{2k\nu^2}$$



The X-ray Luminosity

We showed earlier that the bremsstrahlung emissivity

$$j_\nu(\Omega) = 5.4\times10^{-39} Z^2 n_e n_i T^{-1/2} g_{ff}\, e^{-\frac{h\nu}{kT}} \quad [\mathrm{erg\ cm^{-3}\,s^{-1}\,sr^{-1}\,Hz^{-1}}]$$

In general

$$g_{ff} = g_{ff}(\nu, T)$$

For isotropic emission [sometimes called ε^{ff}]

$$j_\nu = 6.8\times10^{-38} Z^2 n_e n_i T^{-1/2} g_{ff}(\nu,T)\, e^{-\frac{h\nu}{kT}} \quad [\mathrm{erg\ cm^{-3}\,s^{-1}\,Hz^{-1}}]$$

If there is equipartition between electrons and ions, i.e $n_e \approx n_i$ and if $Z\sim1$

$$j_\nu \approx 6.8\times10^{-38}\, n_e^2 T^{-1/2} g_{ff}(\nu,T)\, e^{-\frac{h\nu}{kT}} \quad [\mathrm{erg\ cm^{-3}\,s^{-1}\,Hz^{-1}}]$$

The X-ray luminosity (integrated) will be

$$dL_X = V j_\nu\, d\nu \quad [\text{Prominent X-ray production mechanism}]$$

$$L_X = 6.8\times10^{-38} n_e^2 T^{-1/2} V \int_{\nu_1}^{\nu_2} g_{ff}(\nu,T)\, e^{-\frac{h\nu}{kT}}\, d\nu \quad [\mathrm{erg\ s^{-1}}]$$

where ν_1 and ν_2 represents the upper and lower bounds on the energy band in which the source is observed. This will usually be in the X-ray band. This is why we restrict this analysis to the X-ray spectrum.

The integration can now be performed as follows:

$$L_X = 6.8\times10^{-38}\,\frac{n_e^2 V}{T^{1/2}}\int_{\nu_1}^{\nu_2} g_{ff}(\nu,T)\, e^{-\frac{h\nu}{kT}}\, d\nu$$

we take the value in X-ray band

$$= 6.8\times10^{-38}\,\frac{n_e^2 V}{T^{1/2}}\int_{\nu_1}^{\nu_2} g_X\, e^{-\frac{h\nu}{kT}}\, d\nu$$

$$= 6.8\times10^{-38}\,\frac{n_e^2 V}{T^{1/2}}\, g_X \left[\frac{e^{-\frac{h\nu}{kT}}}{-(h/kT)}\right]_{\nu_1}^{\nu_2}$$

$$= 6.8\times10^{-38}\,\frac{n_e^2 V}{T^{1/2}}\left(\frac{kT}{h}\right) g_X\left[e^{-\frac{h\nu_1}{kT}} - e^{-\frac{h\nu_2}{kT}}\right]$$

$$= 1.4\times10^{-27}\, n_e^2 V\, T^{1/2}\, g_X\left[e^{-\frac{h\nu_1}{kT}} - e^{-\frac{h\nu_2}{kT}}\right]$$

$$= 1.4\times10^{-27}\, n_e^2 V\, T^{1/2}\, f(x)$$

$$f(x) = g_X\left[e^{-\frac{h\nu_1}{kT}} - e^{-\frac{h\nu_2}{kT}}\right]$$

Example: Local Hot Bubble in M-W galaxy

Decoding the Cosmos p. 248

Around the Sun there is an region extending 50-200 pc with temperature $T \sim 10^6$ K with particle density $\sim 10^{-3}$ cm^{-3}. What will be the X-ray Luminosity of this region? If we assume thermal bremsstrahlung:

$$L_X \approx 1.4 \times 10^{-27} n_e^2 V T^{1/2} f(x) \quad \text{erg s}^{-1}$$

Let's say $L \sim 100$ pc

$1\text{pc} = 3.26$ ly
$= 3.26 \times 3 \times 10^{10} \times 86400 \times 365.25$
$\approx 3 \times 10^{18}$ cm

$$V \sim L^3$$
$$\approx (100 \times 3 \times 10^{18} \text{cm})^3$$
$$\approx 2 \times 10^{61} \text{cm}^3$$

$$\Rightarrow L_X \approx 2.8 \times 10^{31} \left(\frac{n_e}{10^{-3}\text{cm}^{-3}}\right)^2 \left(\frac{V}{2\times10^{61}\text{cm}^3}\right)\left(\frac{T}{10^6\text{K}}\right)^{1/2} f(x) \quad \text{erg s}^{-1}$$

The function $f(x)$ has roughly the following value:

For: $E_2 \sim 2$ keV and $E_1 \sim 0.3$ keV $\Rightarrow g_X \sim 0.7$

$$f(x) \approx 1.6$$

$$L_X[0.3\,\text{keV} - 2\,\text{keV}] = 4.5 \times 10^{31} \left(\frac{n_e}{10^{-3}\text{cm}^{-3}}\right)\left(\frac{V}{2\times10^{61}\text{cm}^3}\right)\left(\frac{T}{10^6\text{K}}\right)^{1/2} \times \left(\frac{f(x)}{1.6}\right) \quad (\text{erg s}^{-1})$$

Notice:

$$L_X \ll L_\odot = 4 \times 10^{33} \text{ erg s}^{-1}$$

Bremsstrahlung emissivity and Brightness

The emission coefficient j_ν tells us the emission properties of the medium. To get the full brightness I_ν we need to know the absorption properties of the medium, i.e we need to know α_ν.

Since thermal bremsstrahlung is a collisional process we can use local thermodynamic equilibrium conditions (LTE), where the brightness

$$S_\nu = \frac{j_\nu^{ff}}{\alpha_\nu} \qquad \text{[source function]}$$

For a BB emitter

$$S_\nu = I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

Therefore

$$\alpha_\nu = \frac{j_\nu^{ff}}{I_\nu}$$

$$= \frac{5.4\times10^{-39}\, n_e n_i\, T^{-1/2} Z^2\, g_{ff}(\nu, T)\, e^{-\frac{h\nu}{kT}}}{\frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}}$$

$$= 3.7\times10^{8}\, n_e n_i\, \nu^{-3}\, T^{-1/2} Z^2\, g_{ff} \left(1 - e^{-\frac{h\nu}{kT}}\right)$$

The optical depth can be determined from this

$$\tau_\nu = \int \alpha_\nu\, dl$$

This may vary along line of sight.

$$\approx 3.7\times10^{8} \times \nu^{-3} T^{-1/2} g_{ff}\left(1 - e^{-\frac{h\nu}{kT}}\right) \int n_e n_i\, dl$$

The integral is called the emission measure

$$EM = \int n_e n_i \, dl$$
$$\approx \int n_e^2 \, dl \quad \approx n_e^2 l$$

Hence

$$\tau_\nu = 3.7\times10^8 \, T^{-1/2} Z^2 \nu^{-3} g_{ff}(\nu,T)[1-e^{-\frac{h\nu}{kT}}] \quad EM$$

This emission measure contains information about the number of particles along the line of sight

Since there is very little extinction for frequencies $h\nu \gg kT$, extinction plays a role at $h\nu \ll kT$. Then

$$1-e^{-\frac{h\nu}{kT}} = 1-(1-\frac{h\nu}{kT} \cdots)$$
$$= \frac{h\nu}{kT}$$

$$\tau_\nu = 3.7\times10^8 \, T^{-1/2} Z^2 \nu^{-3} g_{ff}(\nu,T) \left(\frac{h\nu}{kT}\right) \, EM$$
$$= 0.017 \, T^{-3/2} Z^2 \nu^{-2} g_{ff}(\nu,T) \, EM$$

The opacity of source increases at low frequencies

$$\tau_\nu \propto \nu^{-2}$$

Notice that

$$j_\nu \propto \frac{n_e n_i Z^2}{T^{1/2}} e^{-\frac{h\nu}{kT}}$$

For $h\nu \ll kT$ $\quad e^{\frac{h\nu}{kT}} \to 1$

$$j_\nu \propto \frac{n_e n_i Z^2}{T^{1/2}} \times \nu^0$$

NB: we ignored the frequency dependence of g_{ff}, which is weak justifying $\sim \nu^0$.

Hence for $h\nu \ll kT$ the emissivity is constant in terms of frequency

$$I_\nu = \frac{j_\nu}{\alpha_\nu}$$

$$= \frac{j_\nu l}{\alpha_\nu l}$$

$$= \frac{j_\nu l}{\tau_\nu} \qquad \tau_\nu \propto \nu^{-2} T^{-3/2}$$

$$I_\nu \propto \frac{\left(\frac{n_e n_i Z^2}{T^{1/2}}\right) l}{\nu^{-2} T^{-3/2}}$$

$$\propto n_e n_i \, l \, T \, \nu^2$$

$$\propto EM \, T \, \nu^2$$

$$I_\nu \propto \nu^2 \qquad \text{Rayleigh - Jeans}$$

See fig 8.3 p.241 Decoding the Cosmos

H II - Regions

Regions of ionized gas such as H II regions, i.e regions associated with star formation (see Fig 8.4 p 241 of Decoding the Cosmos) emit by thermal bremsstrahlung. The properties of these regions can be derived through radio observations. Usually the temperature is of the order of $T \sim 10^4$ K, not high enough to produce X-ray emission. Therefore they emit at lower energies, i.e radio - optical. We can probe these H II regions by using radio telescopes because radio waves are not absorbed by interstellar dust

Why: We showed earlier that the energy per unit frequency of a dipole (R & L p.88 eq 3.26)

$$\frac{dE}{d\omega} \propto \omega^4$$

$$\propto \left(\frac{1}{\lambda}\right)^4$$

Short wavelengths are scattered more, that is why optical suffer from interstellar extinction and radio waves not !!

NB : Therefore radio observations can provide information about a source or region without having to make assumptions or uncertain corrections for extinction

Also H II regions are usually imbedded in molecular clouds and these regions cannot be observed with optical telescopes and radio - IR observations is the only way to study them (see Fig 9.6 Decoding the Cosmos p. 299).

For any H II region at radio wavelengths

$$h\nu \ll kT$$

Radio : $\nu_{max} \approx 100$ GHz
H II : $T \approx 10^4$ K

$$\Rightarrow \quad E_{radio} \approx h\nu$$
$$\approx 4\times10^{-4} \left(\frac{\nu}{100\,\mathrm{GHz}}\right) \;\mathrm{eV}$$

$$\Rightarrow \quad kT \approx 1\;\mathrm{eV}\left(\frac{T}{10^4\,\mathrm{K}}\right) \qquad \boxed{\alpha \approx \frac{h\nu}{kT} \approx 4\times10^{-4}}$$

Hence in radio wavelengths $h\nu \ll kT$. Therefore we are in the R-J limit

$$I_\nu = \frac{2\nu^2}{c^2}\, kT$$

$$\propto \nu^2 \qquad (\tau_\nu \gg 1)$$

For the optically thin emission we require a value for the Gaunt factor. When $T < 9\times10^5$ K, applicable to H II regions

$$g_{eff}(\nu, T) = 11.962\; T^{0.15}\, \nu^{-0.1}$$

Since the frequency dependence of the optically thin part follows that of the gaunt factor (usually we assume it to be constant) one can see that it is nearly constant

Therefore the optical depth that we gave earlier

$$\tau_\nu = 3.7\times10^{8}\, T^{-1/2} Z^2 \nu^{-3} g_{ff}(\nu,T)\,[1-e^{-\frac{h\nu}{kT}}]\, EM$$

$$= 3.7\times10^{8}\, T^{-1/2} Z^2 \nu^{-3} \times [11.96\, T^{0.15} \nu^{-0.1}] \times$$

$$[1-e^{-\frac{h\nu}{kT}}]\, EM \qquad h\nu \ll kT$$

$$= (3.7\times10^{8})(11.96)(T^{-1/2}T^{0.15})\, Z^2 (\nu^{-3}\nu^{-0.1})\, EM$$

$$(1-(1-\tfrac{h\nu}{kT} \ldots))$$

$$\approx 0.2\,(T^{-1/2}T^{0.15}T^{-1})(\nu^{-3}\nu^{-0.1}\nu)\, Z^2\, EM$$

$$\approx 0.2\, Z^2 T^{-1.35} \nu^{-2.1}\, EM \qquad EM$$

$$\approx 0.2\, Z^2\, T^{-1.35}\, \nu^{-2.1}\, n_e^2 \ell$$

$$\approx 8\times10^{-2}\left(\frac{T}{K}\right)^{-1.35}\left(\frac{\nu}{GHz}\right)^{-2.1}\left(\frac{EM}{pc\ cm^{-6}}\right)$$

This allows us to determine the brightness temp by using the following approximation for $\tau_\nu \ll 1$. This eqn applies to the R-J limit ($h\nu \ll kT$):

$$T_{B\nu} = T(1-e^{-\tau_\nu}) \qquad h\nu \ll kT$$

$$= T(1-(1-\tau_\nu \ldots)) \qquad \tau_\nu \ll 1$$

$$= T\tau_\nu$$

Let's first motivate this by paging back to Chapt 1 of R & L.

We showed earlier that the brightness of a source

$$I_\nu(\tau) = I_\nu(\tau_\nu=0)\, e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

$$= S_\nu(1 - e^{-\tau_\nu})$$ no background source

In the limit $h\nu \ll kT$ the R-J law applies

$$I_\nu = \frac{2\nu^2}{c^2} kT$$

$$\frac{2\nu^2}{c^2} kT_{b\nu} = \frac{2\nu^2}{c^2} kT(1 - e^{-\tau_\nu})$$

$$T_{b\nu} = T(1 - e^{-\tau_\nu})$$

Hence for

$\tau_\nu \gg 1$ $\quad T_{b\nu} \approx T$ $\quad (e^{-\tau_\nu} \to 0)$

$\tau_\nu \ll 1$ $\quad T_{b\nu} \approx T\tau_\nu$ $\quad 1 - e^{-\tau_\nu} = 1 - (1 - \tau_\nu \ldots)$
$\approx \tau_\nu$

For $\tau_\nu \gg 1$ the brightness temp matches the source temp and the source is a BB with no to very little absorption.

For $\tau_\nu \ll 1$ the brightness temp is much lower than the physical temp. This implies that the physical temperature of an optically thin plasma must be higher than the brightness temp of a BB to be at the same brightness.

Now the brightness temp can be estimated

$$T_{b\nu} \approx T \tau_\nu \qquad [\text{for } \tau_\nu \ll 1]$$

$$= T \times 8.2\times10^{-2} \left(\frac{T}{K}\right)^{-1.35} \left(\frac{\nu}{GHz}\right)^{-2.1} \left(\frac{EM}{pc\,cm^{-6}}\right)$$

$$\approx 8.2\times10^{-2} \left(\frac{T}{K}\right)^{-0.35} \left(\frac{\nu}{GHz}\right)^{-2.1} \left(\frac{EM}{pc\,cm^{-6}}\right)$$

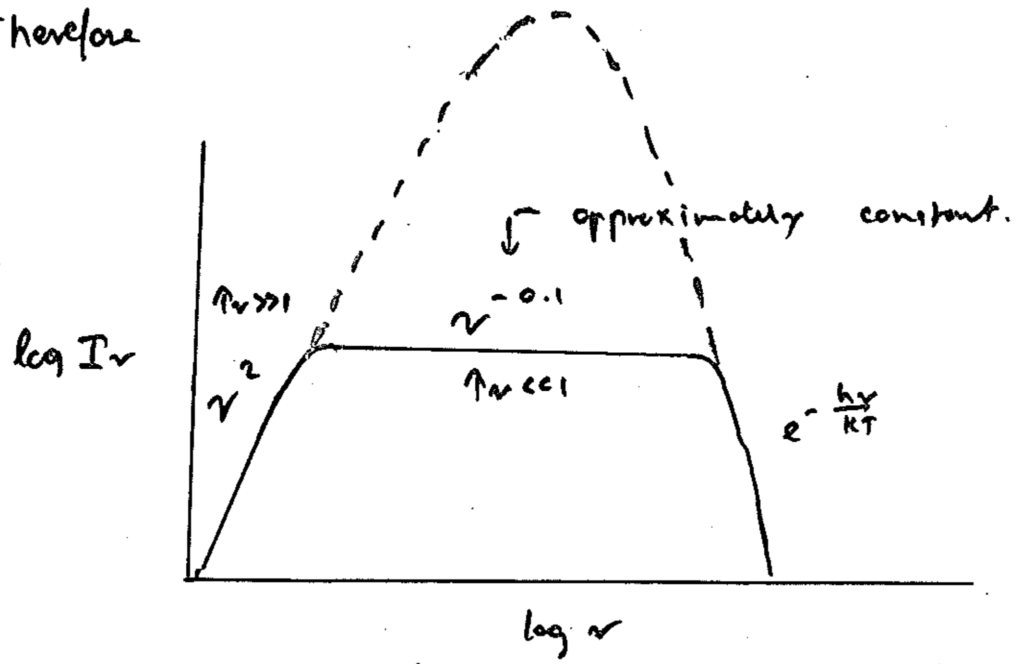
Therefore one can see that for $\tau_\nu \ll 1$

$$I_\nu = \frac{2\nu^2}{c^2} k T_{\nu b} \qquad (h\nu < kT)$$

$$\propto (\nu^2)(\nu^{-2.1})$$

$$\propto \nu^{-0.1} \qquad (\tau_\nu \ll 1)$$

Therefore



See Problem 8.8 p 276 (Decoding the Cosmos)

Example: Accreting White Dwarf.

A hydrogen plasma from a companion star accretes $10^{-9} M_\odot \, yr^{-1}$ onto the surface of a WD of radius 8000 km and mass $0.5 M_\odot$. The plasma is guided to the surface by the magnetic field so that it settles down in a region only $\sim 1\%$ of the WD surface. This layer forms a ~ 1 m optically thin slab that radiates through the bremsstrahlung process. The plasma density in this slab is roughly $\rho \sim 10^{-2}$ kg m^{-3}

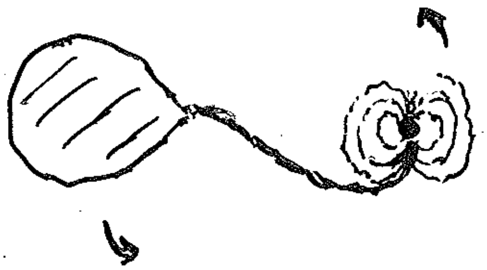
a.) What is the ion (proton) density in this region?

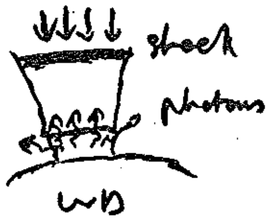
b.) Calculate the PE of the infalling gas lost per second as the material plunges to the WD surface.

c.) If this energy is converted to heat, calculate the energy generation rate (Heat energy).

d.) Determine the steady state equilibrium temperature

e.) Determine the average photon energy if the radiation and plasma is in equilibrium.





$R_* = 8000\ \text{km}$
$= 8\times10^{8}\ \text{cm}$

$M_* = 0.5\ M_\odot$
$= 10^{33}\ \text{g}$

$\dot{m} \approx 10^{-9}\ M_\odot\ \text{yr}^{-1}$
$\approx 6.3\times10^{16}\ \text{g}\cdot\text{s}^{-1}$

a.) The plasma settles in a region onto the pole with area ~ 1% of the WD surface area

$$A_{plasma} = 0.01\ A_{WD}$$

$$\pi R_{pole}^2 \approx 0.01\,(4\pi R_*^2)$$

$$\approx 8\times10^{16}\ \text{cm}^2$$

$$\Rightarrow R_{pole} = 1.6\times10^{8}\ \text{cm}$$

This is the radius of the slab of emitting plasma at the polar cap of the WD. The volume is

$$V_{plasma} = A_{plasma}\times h$$
$$\approx (8\times10^{16}\ \text{cm}^2)(100\ \text{cm})$$
$$\approx 8\times10^{18}\ \text{cm}^3$$

The plasma density is

$$\rho_p = 10^{-2}\ \text{kg m}^{-3} = 10^{-5}\ \text{g cm}^{-3}$$

For a fully ionized plasma the density of ions and electrons is ~ equal

$$n_i = \left(\frac{\rho_e}{m_p}\right) \qquad m_p \approx 1.67 \times 10^{-24}\ g$$

$$\approx 6 \times 10^{18}\ cm^{-3}$$

$$n_e = n_i \approx 6 \times 10^{18}\ cm^{-3}$$

b) The PE lost by the infalling matter which is converted to heat

$$\dot{E}_{grav} = G\frac{M_* \dot{m}}{R_*}$$

$$\approx 5.3 \times 10^{33} \left(\frac{M}{0.5 M_\odot}\right)\left(\frac{\dot{m}}{6 \times 10^{16}\ g\,s^{-1}}\right)\left(\frac{R_*}{8 \times 10^{8}\ cm}\right)^{-1} erg\ s^{-1}$$

$$\Rightarrow \dot{E}_{grav} \approx 5 \times 10^{27}\ Watt \qquad [1\ J = 10^7\ erg]$$

c.) This energy is converted to thermal energy in the shocked region. The power deposited per unit volume is

$$\dot{U}_{heat} = \left(\frac{\dot{E}_{grav}}{V_{plasma}}\right)$$

$$= \frac{5.3 \times 10^{33} \left(\frac{M_*}{0.5 M_\odot}\right)\left(\frac{\dot{m}}{6 \times 10^{16}\ g\,s^{-1}}\right)\left(\frac{R_*}{8 \times 10^{8}\ cm}\right)^{-1} erg\ s^{-1}}{8 \times 10^{18}\ cm^{3}}$$

$$\approx 6.63 \times 10^{14}\ erg\ cm^{-3}\ s^{-1} \left(\frac{M_*}{0.5 M_\odot}\right)\left(\frac{\dot{m}}{6 \times 10^{16}\ g\,s^{-1}}\right) \times \left(\frac{R_*}{8 \times 10^{8}\ cm}\right)^{-4}$$

d.) In a steady state scenario the radiated power per unit volume should balance the conversion rate (to heat) per unit volume. This allows us to estimate the equilibrium temperature

$$\varepsilon^{ff} = 4\pi j(\Omega)$$
$$= 1.42\times10^{-27} n_e n_i Z^2 \hat{g}_{ff} T^{1/2} \text{ erg cm}^{-3}\text{ s}^{-1}$$

Then

$$\dot{U}_{Heat} = \varepsilon_{ff} \quad \text{(steady state)}$$

$$6.63\times10^{14} \text{ erg cm}^{-3}\text{ s}^{-1} = 1.42\times10^{-27} n_e n_i Z^2 \bar{g}_{ff} T^{1/2} \text{ erg cm}^{-3}\text{ s}^{-1}$$

$$\Rightarrow T^{1/2} = \frac{6.63\times10^{14}}{1.42\times10^{-27} n_e n_i Z^2 \bar{g}_{ff}}$$

$Z = 1$ $\qquad n_e = n_i = 10^{19}\text{ cm}^{-3}$

$\hat{g}_{ff} \approx 1.2$

$$T^{1/2} \approx 10^4 \text{ K}$$

$$\Rightarrow T \approx 10^8 \text{ K}$$

d.) $\langle \varepsilon_\gamma \rangle \approx kT$

$$\approx \left(1.38\times10^{-23}\text{ J K}^{-1}\right)\left(10^8\text{ K}\right)$$

$$\approx 1.4\times10^{-15}\text{ J}$$

$$\approx 9000\text{ eV}$$

$$\sim 1\text{ keV} \quad \text{(X-ray emission)}$$