Compton and Inverse-Compton Scattering

The simplest interaction between photons and electrons is scattering. When the energy of the incoming photon is small compared to the electron rest mass energy the process is called Thomson scattering. This can be treated within the framework of classical electrodynamics. If the energy of the incoming photons is comparable to m_0c^2, a quantum treatment is required.

Thomson Cross-Section

Assume the electron is at rest and an e-m wave is impinging with frequency

$$h\nu \ll m_0 c^2$$

$$\nu \ll \frac{m_0 c^2}{h}$$

Assume the incoming wave is linearly polarized. In order to neglect the magnetic Lorentz force associated with the interaction between the oscillating e^- and the B-field of the incoming wave,

the oscillation speed $v \ll c$

$$\boxed{F_m = \frac{e}{c}(\bar{v} \times \bar{B})}$$

oscillation velocity of e^-

$$\bar{F}_m = e(\bar{\beta} \times \bar{B})$$

$$\Rightarrow \beta \to 0$$

The electron starts to oscillate in response to the fluctuating $\bar{E}$-fields associated with the incoming wave.

The average square acceleration during one cycle of duration $T = \frac{1}{\nu}$ is :

$$\langle a^2 \rangle = \frac{1}{T}\int_0^T a^2(t)\,dt$$

$$= \frac{1}{T}\int_0^T \left[\frac{e}{m_0}E(t)\right]^2 dt$$

$$= \frac{1}{T}\frac{e^2}{m_0^2}\int_0^T E^2(t)\,dt$$

$$= \frac{1}{T}\frac{e^2}{m_0^2}\int_0^T E_0^2 \sin^2(2\pi\nu t)\,dt$$

$$= \frac{e^2 E_0^2}{T m_0^2}\int_0^T \sin^2(2\pi\nu t)\,dt$$

$$= \frac{e^2 E_0^2}{m_0^2 T}\left[\frac{t}{2} - \frac{\sin 2(2\pi\nu)t}{4(2\pi\nu)}\right]_0^T$$

$$= \frac{e^2 E_0^2}{m_0^2 T}\left[\frac{T}{2} - \frac{\sin(4\pi\nu T)}{8\pi\nu}\right]$$

$$= \frac{e^2 E_0^2}{2 m_0^2}\left[1 - \frac{2}{8\pi\nu T}\sin(4\pi\nu T)\right]$$

$\sin(4\pi) = 0$
$\nu = \frac{1}{T}$

$$= \frac{e^2 E_0^2}{2 m_0^2}$$

$$\boxed{\begin{aligned} \bar{E}(t) &= \bar{E}_0 \sin(\omega t) \\ &= \bar{E}_0 \sin(2\pi\nu t) \\ a(t) &= \frac{F(t)}{m_0} \\ &= \frac{e\,E(t)}{m_0} \end{aligned}}$$

The emitted power per unit solid angle is given by the Larmor eqn

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \sin^2\Theta}{4\pi c^3}$$ [Eq. 3.18 R&L 79]

Here Θ represents the angle between the acceleration vector and the emitted radiation. Note that Θ is not the angle between the incoming and scattered photon !!

Then

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \sin^2\theta}{4\pi c^3}$$

Substituting in the average acceleration

$$\langle a(t)^2 \rangle = \frac{e^2 E_0^2}{2 m_0^2}$$

$$\frac{dP}{d\Omega} = \frac{e^2 \left(\frac{e^2 E_0^2}{2 m_0^2}\right) \sin^2\theta}{4\pi c^3}$$

$$= \frac{e^4 E_0^2 \sin^2\theta}{8\pi m_0^2 c^3}$$

The scattered radiation is completely linearly polarized in the plane defined by the incident polarization vector of the incoming photon and the resultant scattered photon. The flux of the incoming radiation is

$$S_i = \frac{c}{8\pi} E_0^2$$

The differential cross-section for this scattering is

$$\left(\frac{d\sigma}{d\Omega}\right)_{pol} = \frac{\frac{dP}{d\Omega}}{S_i}$$

$$= \frac{\left[\frac{e^4 E_0^2 \sin^2\theta}{8\pi m_0^2 c^3}\right]}{\left[\frac{c}{8\pi} E_0^2\right]}$$

$$= \left(\frac{e^4}{m_e^2 c^4}\right) \sin^2\theta$$

$$= r_0^2 \sin^2\theta$$

$$\boxed{r_0 = \frac{e^2}{m_e c^2} = 2.82 \times 10^{-13}\,\text{cm}}$$

The total scattered power is

$$P = \int \frac{dP}{d\Omega}\, d\Omega$$

$$= \frac{e^4 E_0^2}{8\pi m_0^2 c^3} \int_{\Omega} \sin^2\Theta \sin\Theta \, d\Theta \, d\phi$$

$$= \frac{2\pi e^4 E_0^2}{8\pi m_0^2 c^3} \int_0^{\pi} \sin^3\Theta \, d\Theta$$

$$= \frac{e^4 E_0^2}{4 m_0^2 c^3} \left[-\cos\Theta + \frac{\cos^3\Theta}{3} \right]_0^{\pi}$$

$$= \frac{e^4 E_0^2}{3 m_e^2 c^3}$$

The total cross-section:

$$\sigma_{pd} = \frac{P}{S_i}$$

$$= \left[\frac{e^4 E_0^2}{3 m_e^2 c^3}\right] \Big/ \left[\frac{c E_0^2}{8\pi}\right]$$

$$= \underbrace{\frac{8\pi}{3} r_0^2}_{\text{Thomson cross-section}} \qquad r_0^2 = \frac{e^4}{m_e^2 c^4}$$

↳ Thomson cross. section

The Klein - Nishina Cross Section

The Thomson cross section is just the classical limit of the more general cross section proposed by Klein-Nishina from a rigorous QED analysis

$$\frac{d\sigma_{K\cdot N}}{d\Omega} = \frac{3}{16\pi}\,\sigma_T\left(\frac{x_1}{x}\right)^2\left[\frac{x}{x_1} - \frac{x_1}{x} - \sin^2\theta\right]$$

The full expression is

$$\frac{d\sigma_{K\cdot N}}{d\Omega} = \frac{3}{16\pi}\,\frac{\sigma_T}{\left[1+x(1-\cos\theta)\right]^2}\left[x(1-\cos\theta) + \frac{1}{1+x(1-\cos\theta)} + \cos^2\theta\right]$$

Notice here that

$$\underbrace{x = \left(\frac{\varepsilon}{m_0c^2}\right)}_{\text{Energy of incoming photon}} \qquad \underbrace{x_1 = \frac{\varepsilon_1}{m_0c^2}}_{\text{Energy of scattered photon}}$$

In the form immediately above the differential cross section is only dependent on x, the energy of the incoming photon. For $\theta \to 0$

$$\frac{d\sigma_{K\cdot N}}{d\Omega} \to \frac{3}{16\pi}\,\underbrace{\sigma_T}_{\frac{8\pi}{3}r_0^2}\left[0 + 1 + 1\right]$$

$$\to \frac{3}{8\pi}\,\sigma_T \quad \text{[Thomson limit where Compton scattering is elastic]}$$

$$\to r_0^2$$

Notice that for $\theta \to 0$

$$\left(\frac{x_1}{x}\right) = \frac{1}{1+x(1-\cos\theta)} \to \frac{x_1}{x} \to 1 \quad \text{Elastic scattering}$$

This is equivalent to

$$\left(\frac{d\sigma_T}{d\Omega}\right)_{unpol} = \frac{1}{2} r_0^2 (1 + \cos^2\theta)$$

$$= r_0^2 \qquad \theta \to 0$$

This is an extremely small value $\sim 10^{-24}\ cm^2$.

The total cross-section in K-N limit is

$$\sigma_{KN} \sim \sigma_T \left[1 - 2x + \frac{26x^2}{5} + \dots\right] \qquad x \ll 1$$

$$\sigma_{KN} \sim \frac{3}{8}\left(\frac{\sigma_T}{x}\right)\left[\ln(2x) + \frac{1}{2}\right] \qquad x \gg 1$$

Exercise: From the K-N cross-section, i.e

$$\frac{d\sigma_{K\text{-}N}}{d\Omega} = \frac{3}{16\pi}\sigma_T\left(\frac{x_1}{x}\right)\left[\frac{x}{x_1} - \frac{x_1}{x} - \sin^2\theta\right]$$

the total cross-section can be derived

$$\sigma_{K\text{-}N} = \int_\Omega \frac{d\sigma_{K\text{-}N}}{d\Omega}\, d\Omega \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

$$\phi = [0, 2\pi]$$

$$\theta = [0, \pi]$$

which is

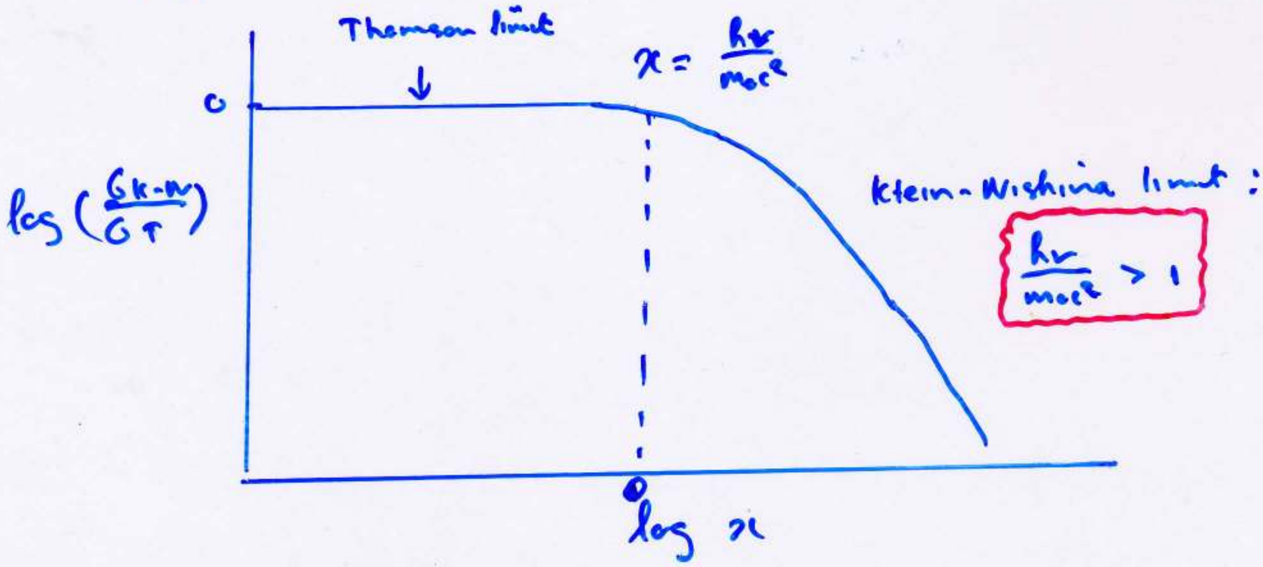
$$\sigma_{K\text{-}N} = \frac{3}{4}\sigma_T\left[\frac{1+x}{x^3}\left[\frac{2x(1+x)}{1+2x} - \ln(1+2x)\right] + \frac{1}{2x}\ln(1+2x) - \frac{(1+3x)}{(1+2x)^2}\right] \qquad \text{R\&L Eq 7.5}$$

$x = \frac{h\nu}{m_0c^2}$ (incoming photon energy)

$x_1 = \frac{h\nu_1}{m_0c^2}$ (scattered photon energy)

$\sigma_T = \frac{8\pi}{3} r_0^2$

Graphically the Klein-Nishina cross-section can be displayed as follows:



Compton scattering is more effective in Thomson limit since the cross-section has a maximum value, i.e σ_T. Scattering between electrons and photons with $x > 1$ is less effective since $\sigma_{K-N} \to 0$ in that limit.

Inverse - Compton Scattering

Inverse - Compton scattering occurs when the electron is not at rest but typically has energies exceeding the photon energy. Therefore in this process energy can be transferred from the electrons to photons. Since electrons can be relativistic, e.g in AGN, Black Hole accretion discs etc, this is a very effective mechanism to produce high energy gamma-rays in these systems.

Inverse - Compton scattering can be evaluated in two regimes, i.e the Thomson limit and the Klein-Nishina limit

Thomson & K-N limits

We transform to a reference frame where the electron is in rest [frame co-moving with e^-] and evaluate the incoming photon energy in that frame.

If :

$$x' = \frac{h\nu'}{m_e c^2} < 1 \quad \Rightarrow \text{Thomson limit}$$

$$x' = \frac{h\nu'}{m_e c^2} > 1 \quad \Rightarrow \text{K-N limit}$$

Very important :

The energy of the incoming photon in the rest frame of the electron will be Doppler boosted

$[\Gamma = \gamma]$

$$\nu' = \frac{\nu}{\Gamma(1 - \beta \cos\theta')} \qquad \cos\theta' \to \beta$$

$$= \frac{\nu}{\Gamma(1-\beta^2)} = \frac{\Gamma^2}{\Gamma}\nu = \Gamma\nu$$

$$= \gamma\nu$$

Therefore in the frame co-moving with electron the incoming photons (within $\theta' \leq \frac{1}{\Gamma}$) will all be Doppler boosted. Therefore for IC scattering we have:

Thomson limit: $x' = \frac{h\nu'}{m_0 c^2} < 1$

$$\Rightarrow \frac{\gamma h\nu}{m_0 c^2} < 1$$

$$\Rightarrow \boxed{h\nu < \frac{m_0 c^2}{\gamma}}$$

K-N limit: $x' = \frac{h\nu'}{m_0 c^2} > 1$

$$\Rightarrow \boxed{h\nu > \frac{m_0 c^2}{\gamma}}$$

This translates to

Thomson: $\nu < \frac{m_0 c^2}{\gamma h}$

K-N limit: $\nu > \frac{m_0 c^2}{\gamma h}$

Example: Consider the IC scattering between optical photons with energy $\epsilon \sim 1\,eV$ and relativistic electrons with Lorentz factor $\gamma \sim 10^4$. In which limit will this scattering occurs?

$E_{phot} = h\nu \Rightarrow \nu = \frac{E_{phot}}{h} \approx 2.4 \times 10^{14} \left(\frac{E_{phot}}{1\,eV}\right) Hz$

$$\Rightarrow \frac{m_0 c^2}{\gamma h} \approx 1.24 \times 10^{16}$$

Therefore:

$$\left.\begin{array}{l} \nu \sim 2.4 \times 10^{14} \\ \Rightarrow \frac{m_0 c^2}{\gamma h} \sim 1.2 \times 10^{16} \end{array}\right\} \nu \ll \frac{m_0 c^2}{\gamma h} \quad \text{(Thomson)}$$

Another Limit

We mentioned earlier that in order to minimize the effect of the magnetic Lorentz force the electron must have a transverse ($\perp$ to incoming wave) velocity much smaller than the speed of light

$$\Rightarrow V_\perp \ll c$$

Consider a incoming wave with oscillating E-field

$$E = E_0 \sin \omega t$$

This implies that from Coulomb's law

$$F_e(t) = e\,E(t)$$

$$M_e a_\perp(t) = e\,E_\perp(t)$$

$$a_\perp(t) = \frac{e}{m_e} E_\perp(t)$$

This implies that the range in the perpendicular velocity

$$dV_\perp = a_\perp \, dt$$

$$\frac{V_\perp}{c} = \frac{1}{c}\int_0^{T/2} \frac{e}{M_e} E_0 \sin \omega t \, dt$$

$$= \frac{eE_0}{m_e c}\int_0^{T/2} \sin \omega t \, dt$$

$$= \frac{eE_0}{M_e c}\left[-\frac{\cos \omega t}{\omega}\right]_0^{T/2} \qquad \omega = \frac{2\pi}{T}$$

$$= \frac{-eE_0}{M_e c\,\omega}\left[(-1)-(1)\right]$$

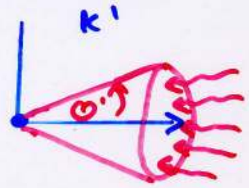
$$= \frac{2eE_0}{M_e c\,\omega} \ll 1$$

$\Rightarrow E_0 \rightarrow$ small

$\Rightarrow \omega$ not too small frequencies

Inverse - Compton : Typical Frequencies

Angle Convention: Chapt 3



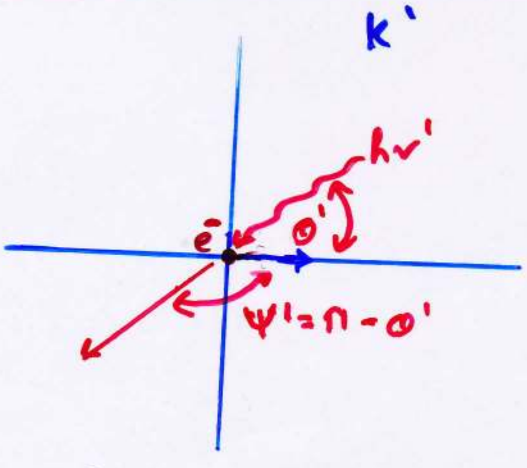
In Chapt 3 we used the following relation for the frequency a co-moving observer in jet (or e^- for that matter) will measure from the incoming photon field

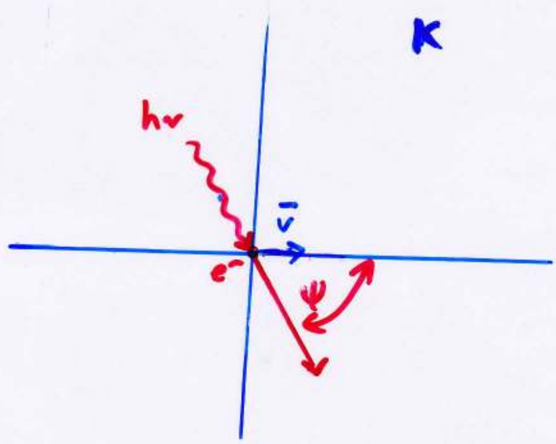
$$\nu'_{obs} = \frac{\nu_{em}}{\Gamma(1-\beta\cos\theta')}$$

$$= \delta'_D\, \nu_{em}$$

To standardize now in terms of a fixed reference frame to evaluate the angles between the incoming photons and relativistic electron which moves in x - direction:

we make the following transformation





The frequency ν'_{obs} in K' can now be written in terms of the abberation formulae:

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta} \qquad \cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'}$$

$$\sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)} \qquad \sin\theta = \frac{\sin\theta'}{\Gamma(1+\beta\cos\theta')}$$

Notice that

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\Rightarrow \cos\theta' = -\cos(\pi - \theta')$$
$$= -[\cos\pi\cos\theta' + \sin\pi\sin\theta']$$

Therefore, in K' the e^- sees

$$\nu' = \frac{\nu_{em}}{\Gamma(1-\beta\cos\theta')}$$

$$= \frac{\nu_{em}}{\Gamma(1+\beta\cos(\pi-\theta'))}$$

$$= \frac{\nu_{em}}{\Gamma(1+\beta\cos\psi')} \qquad \psi' = \pi - \theta'$$

$$= \frac{\nu_{em}}{\Gamma\left(1+\beta\left[\frac{\cos\psi-\beta}{1-\beta\cos\psi}\right]\right)}$$

$$= \frac{\nu_{em}}{\Gamma\left(\frac{1-\beta\cos\psi+\beta\cos\psi-\beta^2}{1-\beta\cos\psi}\right)}$$

$$= \frac{(1-\beta\cos\psi)\,\nu_{em}}{\Gamma(1-\beta^2)}$$

$$= \Gamma(1-\beta\cos\psi)\,\nu_{em} \qquad (1-\beta^2) = \frac{1}{\Gamma^2}$$

↑ ψ is angle between e^- velocity and incoming photon in Lab system K.

This can be written

$$\epsilon' = h\nu' = \Gamma(1-\beta\cos\psi)\,h\nu_{em}$$

$$\left(\frac{h\nu'}{m_0c^2}\right) = \Gamma(1-\beta\cos\psi)\left(\frac{h\nu_{em}}{m_0c^2}\right)$$

$$x' = \Gamma(1-\beta\cos\psi)\,x$$

$$x' = \frac{h\nu'}{m_0c^2}, \quad x = \frac{h\nu}{m_0c^2}$$

Consistency check

From the two equations

$$① \quad \nu' = \frac{\nu_{em}}{\Gamma(1-\beta\cos\theta')}$$

$$② \quad \nu' = \Gamma(1-\beta\cos\psi)\,\nu_{em}$$

One can see that for:

From ①

$$[\text{in } K'] \quad \left.\begin{array}{l} \sin\theta' \sim \frac{1}{\Gamma} \\ \cos\theta' \sim \beta \end{array}\right\} \quad \nu' = \frac{\nu_{em}}{\Gamma(1-\beta^2)} = \Gamma\,\nu_{em}$$

Hence from ②: For $\nu' = \Gamma\,\nu_{em} = \Gamma(1-\beta\cos\psi)\,\nu_{em}$

$$1-\beta\cos\psi = 1$$

$$-\beta\cos\psi = 0$$

$$\cos\psi = 0$$

In K $\Rightarrow$ $\psi = \pi/2$ [as before]

↳ External photons with angles $\psi \to \frac{\pi}{2}$ will have angles $\theta' \sim \frac{1}{\Gamma}$ in the co-moving reference frame.

$\psi \to \frac{\pi}{2}$ [in K] correspond to $\theta' \sim \frac{1}{\Gamma}$ [in K']

For $\theta' \to 0$ (in K') we have the following:

For $\theta' \to 0$ in K'

$$\nu' = \frac{\nu_{em}}{\Gamma(1-\beta)} \qquad \theta' \to 0 \quad \cos\theta' \to 1$$

$$= \frac{(1+\beta)\,\nu_{em}}{\Gamma(1-\beta^2)}$$

$$= \Gamma(1+\beta)\nu_{em}$$

Hence

$$\Gamma(1+\beta)\,\nu_{em} = \Gamma(1-\beta\cos\psi)\,\nu_{em}$$

$$(1+\beta) = (1-\beta\cos\psi)$$

$$\Rightarrow \quad 1+\beta-1+\beta\cos\psi = 0$$

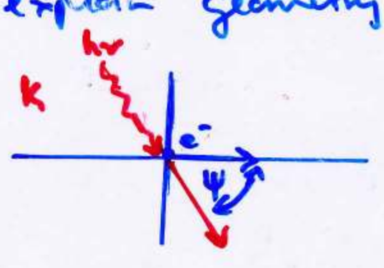
$$\beta + \beta\cos\psi = 0$$

$$\beta(1+\cos\psi) = 0$$

$$\cos\psi = -1$$

$$\psi = 180°$$

These photons interact with the electron right from the front in the lab frame as well, but now just expressed in terms of $\psi \to 180°$ (see Figure to explain geometry)



Scattering Energy: Thomson Regime

A very convenient way to treat Inverse Compton scattering is to evaluate the process in the co-moving reference frame. When we take $\boxed{x' = \frac{h\nu'}{m_0c^2} < 1}$ we are in the Thomson limit and then we know the scattering process is Elastic and

$$\Rightarrow \boxed{x_1' = x'}$$

regardless of angle

Note that

x_1' = scattered photon energy in K'

x' = incoming photon energy in K'

The photon will be scattered at an angle ψ_1' with respect to the electron velocity. One can then use the Doppler eqn's to transform this scattered energy (x_1') and angle (ψ_1') to the Lab frame (K). This is done in the following way:

$$x_1 = \frac{h\nu_1}{m_0c^2} = \delta_D\left(\frac{h\nu_1'}{m_0c^2}\right)$$
$$= \delta_D\, x_1'$$

with

$$\delta_D = \frac{1}{\Gamma(1-\beta\cos\psi_1)}$$

Therefore

$$x_1 = \delta_D \, x_1'$$

$$= \frac{1}{\Gamma(1-\beta\cos\psi_1)} \, x_1'$$

$$= \frac{1}{\Gamma\left(1-\beta\left[\frac{\beta+\cos\psi_1'}{1+\beta\cos\psi_1'}\right]\right)} \, x_1'$$

Here we used the alleration formula

$$\cos\psi_1 = \left[\frac{\beta+\cos\psi_1'}{1+\beta\cos\psi_1'}\right]$$

Therefore

$$x_1 = \frac{x_1'}{\Gamma\left[\frac{1+\beta\cos\psi_1' - \beta^2 - \beta\cos\psi_1'}{1+\beta\cos\psi_1'}\right]}$$

$$= \frac{(1+\beta\cos\psi_1')\, x_1'}{\Gamma\left[\ 1-\beta^2\ \right]}$$

$$= \frac{\Gamma^2(1+\beta\cos\psi_1')\, x_1'}{\Gamma}$$

$$= \Gamma(1+\beta\cos\psi_1')\, x_1'$$

We can also express all quantities in Lab frame:

$$\boxed{x_1' = x' = \Gamma(1-\beta\cos\psi)\, x}$$

$$\underbrace{x' = \Gamma(1-\beta\cos\psi)\, x}$$

derived earlier

This gives

$$\lambda_1 = \gamma_D \, \lambda_1'$$

$$= \frac{1}{\Gamma(1-\beta \cos\psi_1)} \times \lambda'$$

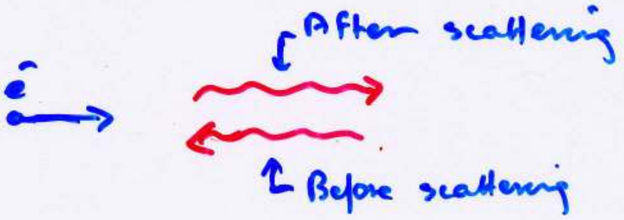
$$= \frac{1}{\Gamma(1-\beta \cos\psi_1)} \times \Gamma(1-\beta\cos\psi)\, \lambda$$

$$= \left(\frac{1-\beta\cos\psi}{1-\beta\cos\psi_1}\right) \lambda$$

Now all quantities are expressed in the Lab frame:

Maximum Energy Transfer

The maximum energy transfer from the electron to the scattered photon is if the collision is head-on and the photon is scattered in opposite direction is from where it came



Incoming photon angle relative to e^- : $\psi = \pi$
Scattered photon angle relative to e^- : $\psi_1 = 0$

$$(\lambda_1)_{max} = \left[\frac{1-\beta\cos(\pi)}{1-\beta\cos(0)}\right] \lambda$$

$$= \left[\frac{1-(-1)\beta}{1-\beta}\right] \lambda$$

$$= \left(\frac{1+\beta}{1-\beta}\right) \lambda$$

This gives

$$(x_1)_{max} = \frac{(1+\beta)}{(1-\beta)}\, x$$

$$= \frac{(1+\beta)^2}{(1-\beta)(1+\beta)}\, x$$

$$= \frac{(1+\beta)^2}{(1-\beta^2)}\, x$$

$$= \Gamma^2 (1+\beta)^2 x$$

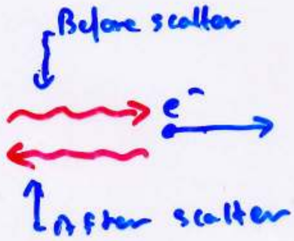
For $\beta \to 1$

$$(x_1)_{max} = 4\,\Gamma^2 x$$

$$= 4\gamma^2 x \qquad (\Gamma = \gamma)$$

Minimum Energy Transfer

The minimum energy transfer from e^- to photon is during a head-tail collision. Here the photon approaches the e^- from behind and scatteres in opposite directions



Incoming photon angle relative to e^- : $\psi = 0$

Scattered photon angle relative to e^- : $\psi_1 = \pi$

$$(x_1)_{min} = \left[\frac{1-\beta\cos(0)}{1-\beta\cos(\pi)}\right] x$$

$$\doteq \left(\frac{1-\beta}{1+\beta}\right) x = \frac{(1-\beta)(1+\beta)}{(1+\beta)^2}\, x = \frac{1}{\Gamma^2(1+\beta)^2}\, x$$

For $\beta \to 1$

$$(x_1)_{min} = \frac{1}{\Gamma^2(1+\beta)^2}\,x = \frac{1}{4\Gamma^2}\,x = \frac{1}{4\gamma^2}\,x \qquad \Gamma = \gamma$$

Summary:

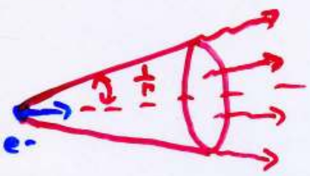
Maximum Energy Transfer:

$$\left.\begin{array}{l}\psi = [\pi] \\ \psi_1 = [0] \\ \beta \to 1\end{array}\right\} \quad x_1 = 4\gamma^2 x$$

Minimum Energy Transfer:

$$\left.\begin{array}{l}\psi = [0] \\ \psi_1 = [\pi] \\ \beta \to 1\end{array}\right\} \quad x_1 = \frac{x}{4\gamma^2}$$

Another case: Scattering into a beaming cone with respect to observer in K



$$\Rightarrow \sin\psi_1 = \frac{1}{\Gamma} = \frac{1}{\gamma}$$

$$\cos\psi_1 = \beta$$

$$\left.\begin{aligned} x_1 &= \left(\frac{1-\beta\cos\psi}{1-\beta\cos\psi_1}\right)x \\ &= \left(\frac{1-\beta\cos\psi}{1-\beta^2}\right)x \\ &= \Gamma^2(1-\beta\cos\psi)x \end{aligned}\right\}$$

$\psi = 0$: $x_1 = \frac{(1-\beta)x}{(1-\beta)(1+\beta)} = \frac{x}{1+\beta}$

$\psi = \frac{\pi}{2}$: $x_1 = \Gamma^2 x = \gamma^2 x$

$\psi = \pi$: $x_1 = \frac{x}{1-\beta}$

Isotropic Photon Distribution

We showed earlier that a co-moving observer in K' see an isotropic energy density of the background photons

$$U' = \frac{4}{3}\Gamma^2 U_{iso} \qquad \beta \to 1$$

If the observer in K' is co-moving with an electron moving with high velocity ($\beta \to 1$) then

$$U' = \frac{N'\langle\varepsilon'\rangle}{V'} \qquad U_{iso} = \frac{N\langle\varepsilon\rangle}{V}$$

$$\frac{N'\langle\varepsilon'\rangle}{V'} = \frac{4}{3}\Gamma^2 \frac{N\langle\varepsilon\rangle}{V}$$

But $N' = N$

$$\frac{\langle\varepsilon'\rangle}{\langle\varepsilon\rangle} = \frac{4}{3}\Gamma^2 \frac{V'}{V}$$

We know that

$$\left.\begin{aligned} V' &= A'\ell' \\ &= A\,\Gamma\ell \\ &= \Gamma A\ell = \Gamma V \end{aligned}\right\} \frac{V'}{V} = \Gamma$$

$$\frac{\langle\varepsilon'\rangle}{\langle\varepsilon\rangle} = \frac{4}{3}\Gamma^2 \times \Gamma$$

In the co-moving frame the electron scatter the photon elastically in the Thomson limit

$$\langle\varepsilon'\rangle = \langle\varepsilon_1'\rangle$$

$$\Rightarrow \frac{\langle\varepsilon_1'\rangle}{\langle\varepsilon\rangle} = \frac{4}{3}\Gamma^2 \times \Gamma$$

$$\frac{\langle \epsilon_1' \rangle}{\langle \epsilon \rangle} = \frac{4}{3} \Gamma^3$$

But the relation between the scattered energy in k' and k is

$$\langle \epsilon_1' \rangle = \Gamma (1 - \beta \langle \cos \psi \rangle) \langle \epsilon_1 \rangle$$

$$= \Gamma \langle \epsilon_1 \rangle \quad \text{since } \langle \cos \psi \rangle = 0$$

$$\Rightarrow \quad \frac{\langle \epsilon_1' \rangle}{\langle \epsilon \rangle} = \Gamma \frac{\langle \epsilon_1 \rangle}{\langle \epsilon \rangle} = \frac{4}{3} \Gamma^3$$

$$\Rightarrow \quad \frac{\langle \epsilon_1 \rangle}{\langle \epsilon \rangle} = \frac{4}{3} \Gamma^2$$

$$\Rightarrow \quad \langle \epsilon_1 \rangle = \frac{4}{3} \Gamma^2 \langle \epsilon \rangle$$

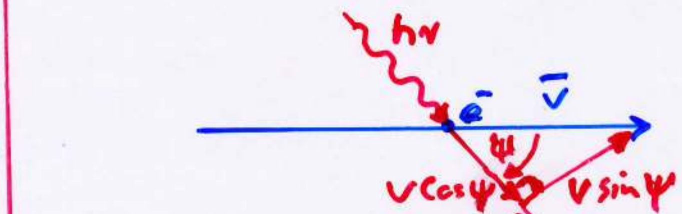
The Total Loss Rate

To do this we simply calculate the rate of scatterings per electron considering all quantities in the Lab. frame.

Let $n(\epsilon)$ be the density of photons of energy $\epsilon = h\nu$ in the Lab frame. Let v represent the electron velocity and ψ (as before) the angle between the electron velocity and the incoming photon. For mono-directional photon distributions we have that the number of interactions or scatterings between electrons and photons per unit time is

$$\frac{dN}{dt} = \int \sigma_T \, V_{rel} \, n(\epsilon) \, d\epsilon$$

Here : $\boxed{V_{rel} = c - v \cos\psi}$



$$\frac{dN}{dt} = \int \sigma_T \left(c - v \cos\psi \right) n(\epsilon) \, d\epsilon$$

$$= \int c \, \sigma_T \left(1 - \beta \cos\psi \right) n(\epsilon) \, d\epsilon \qquad \beta = \frac{v}{c}$$

Note that the number of scatterings between electrons and photons

$$\frac{dN}{dt} \propto (1 - \beta \cos\psi)$$

$$\to 0 \quad \text{for } \psi \to 0 \text{ and } \beta \to 1$$

So it implicitly mean that the optical depth will be greatly reduced if photons and electrons move in the same direction.

The power contained in scattered radiation is

$$\frac{dE_\gamma}{dt} = \underbrace{\varepsilon_1}_{\text{Energy of scattered photon}} \overbrace{\frac{dN}{dt}}^{\text{scattering rate.}}$$

Use the notation

$$x_1 = \frac{\varepsilon_1}{m_e c^2}$$

$$x = \frac{\varepsilon}{m_e c^2}$$

we showed earlier that

$$x_1 = \frac{\varepsilon_1}{m_e c^2} = x\left[\frac{1 - \beta \cos\psi}{1 - \beta \cos\psi_1}\right]$$

$$= \frac{\varepsilon}{m_e c^2}\left[\frac{1 - \beta \cos\psi}{1 - \beta \cos\psi_1}\right]$$

$$\Rightarrow \varepsilon_1 = \varepsilon \left(\frac{1 - \beta \cos\psi}{1 - \beta \cos\psi_1}\right)$$

$$\frac{dE_\gamma}{dt} = \epsilon_1 \frac{dN}{dt}$$

$$= \int \epsilon_1 c\sigma_T (1-\beta\cos\psi)\, n(\epsilon)\, d\epsilon$$

$$= \int c\sigma_T\, \epsilon \frac{(1-\beta\cos\psi)}{(1-\beta\cos\psi_1)} (1-\beta\cos\psi)\, n(\epsilon)\, d\epsilon$$

$$= c\sigma_T \int \epsilon \frac{(1-\beta\cos\psi)^2}{(1-\beta\cos\psi_1)}\, n(\epsilon)\, d\epsilon$$

From the vantage point of the observer in K the photons interact with the electron isotropically, but they are scattered into a forward cone with half-angle

$$\sin\psi_1 = \frac{1}{\gamma}$$

This corresponds to

$$\sin^2\psi_1 = \frac{1}{\gamma^2}$$

$$(1-\cos^2\psi_1) = \frac{1}{\gamma^2}$$

$$\Rightarrow \cos^2\psi_1 = \beta^2$$

$$\Rightarrow \cos\psi_1 = \beta \qquad \text{[showed earlier]}$$

Therefore, for high energy electrons

$$\frac{dE_\gamma}{dt} = c\sigma_T \int \frac{(1-\beta\cos\psi)^2}{(1-\beta^2)}\, \epsilon\, n(\epsilon)\, d\epsilon$$

$$= c\sigma_T \gamma^2 \int (1-\beta\cos\psi)^2\, \epsilon\, n(\epsilon)\, d\epsilon$$

The spatial average high energy photon production rate

$$\left\langle \frac{d\mathcal{E}_\gamma}{dt} \right\rangle = c\,\sigma_T\,\gamma^2 \int \langle (1-\beta\cos\psi)^2 \rangle\, \epsilon\, n(\epsilon)\, d\epsilon$$

Notice that

$$\langle (1-\beta\cos\psi)^2 \rangle = \langle 1 - 2\beta\cos\psi + \beta^2\cos^2\psi \rangle$$

$$= 1 - 2\beta\langle\cos\psi\rangle + \beta^2\langle\cos^2\psi\rangle$$

Exercise: Show that

$$\langle \cos\psi \rangle = 0$$

$$\langle \cos^2\psi \rangle = \frac{1}{3}$$

Hint:

$$\langle \cos\psi \rangle = \frac{\int \cos\psi\, d\Omega}{d\Omega}$$

$$\langle \cos^2\psi \rangle = \frac{\int \cos^2\psi\, d\Omega}{d\Omega}$$

Therefore

$$\langle (1-\beta\cos\psi)^2 \rangle = 1 + \frac{\beta^2}{3}$$

Hence:

$$\left\langle \frac{d\mathcal{E}_\gamma}{dt} \right\rangle = c\,\sigma_T\,\gamma^2 \int \left(1 + \frac{\beta^2}{3}\right) \epsilon\, n(\epsilon)\, d\epsilon$$

$$= \left(1 + \frac{\beta^2}{3}\right) c\,\sigma_T\,\gamma^2 \int \epsilon\, n(\epsilon)\, d\epsilon$$

Hence

$$\left\langle \frac{d\mathcal{E}_\gamma}{dt} \right\rangle = \left(1+\frac{\beta^2}{3}\right)\gamma^2 c\,\sigma_T\, U_{ph}$$

$$= \left(1+\frac{\beta^2}{3}\right)\gamma^2 c\,\sigma_T\,\overbrace{\varepsilon_{ph}\, n_{ph}}^{U_{ph}}$$

$U_{ph} = \int \varepsilon\, n(\varepsilon)\, d\varepsilon$ Total energy density of background photon field

$$= \left[\left(1+\frac{\beta^2}{3}\right)\gamma^2 \varepsilon_{ph}\right] \times \Big[\underbrace{c\,\sigma_T\, n_{ph}}_{[s^{-1}]}\Big]$$

$$= \Big[\underbrace{\tfrac{4}{3}\gamma^2 \varepsilon_{ph}}_{\langle \varepsilon_1 \rangle}\Big] \times \underbrace{\text{Rate}}_{[s^{-1}]}$$

$$\Rightarrow \quad \langle \varepsilon_1 \rangle = \tfrac{4}{3}\gamma^2 \varepsilon_{ph}$$ showed earlier

This is the power contained in scattered radiation. This is added on top of the already existing background photon field. Therefore to probe "only" the IC power we subtract the background field

$$P_c(\gamma) = \left\langle \frac{d\mathcal{E}_\gamma}{dt} \right\rangle - \overbrace{\sigma_T\, c\, U_{ph}}^{\text{background photon field}}$$

$$= \left(1+\frac{\beta^2}{3}\right)\gamma^2 c\,\sigma_T\, U_{ph} - \sigma_T\, c\, U_{ph}$$

$$= \gamma^2 c\,\sigma_T\, U_{ph} + \frac{\gamma^2\beta^2}{3} c\,\sigma_T\, U_{ph} - \sigma_T\, c\, U_{ph}$$

$$= \frac{c\,\sigma_T\, U_{ph}}{1-\beta^2} + \frac{\frac{1}{3}\beta^2 c\,\sigma_T\, U_{ph}}{(1-\beta^2)} - \frac{(1-\beta^2)}{(1-\beta^2)}\sigma_T\, c\, U_{ph}$$

$$= \frac{c\,\sigma_T\, U_{ph} + \frac{1}{3}\beta^2 c\,\sigma_T\, U_{ph} - \sigma_T\, c\, U_{ph} + \beta^2 \sigma_T\, c\, U_{ph}}{1-\beta^2}$$

$$= \frac{\left(1+\frac{1}{3}\right)\beta^2 c\,\sigma_T\, U_{ph}}{1-\beta^2}$$

$$= \frac{4}{3}\gamma^2\beta^2 c\,\sigma_T\, U_{ph}$$

$$= \Big[\underbrace{\tfrac{4}{3}\gamma^2 \varepsilon_{ph}}_{\langle \varepsilon_1 \rangle}\Big] \times \Big[\underbrace{c\,\sigma_T\, n_{ph}}_{\langle \text{rate} \rangle}\Big] \qquad \beta \to 1$$

Notice the similarity

- $P_c(\gamma) = \frac{4}{3} \sigma_T c \gamma^2 \beta U_{ph}$
- $P_{syn}(\gamma) = \frac{4}{3} \sigma_T c \gamma^2 \beta U_{mag}$

$$\alpha = \frac{P_{syn}(\gamma)}{P_c(\gamma)} \propto \frac{U_{mag}}{U_{ph}}$$

However at low frequencies electrons can radiate and absorb synchrotron radiation. So synchrotron cooling of relativistic electrons can also manifest in heating of the thermal plasma

One can now investigate the IC scattering between relativistic electrons and anisotropic photon distributions. In the first case the photons will come mainly from the front and back the following case, from the sides.

The γ-ray production rate

$$\frac{dE_\gamma}{dt} = c\sigma_T \gamma^2 (1-\beta \cos\psi)^2 \int \varepsilon n(\varepsilon) d\varepsilon$$

$$= \left[c\sigma_T \gamma^2 \int \varepsilon n(\varepsilon) d\varepsilon\right] (1-\beta\cos\psi)^2$$

$$\Rightarrow \frac{dE_\gamma(\cos\psi)}{dt} = \text{Const}\,(1-\beta\cos\psi)^2$$

$$\Rightarrow P_\gamma(\mu) = \text{Const}\,(1-\beta\mu)^2 \qquad \mu = \cos\psi$$

in Lab system.

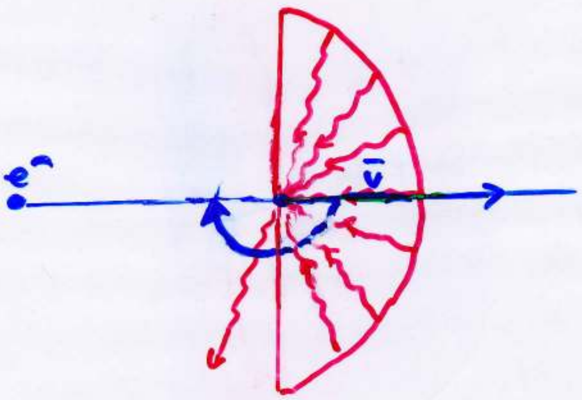
To investigate the integrated angular dependence over a range of angles

$$\int \rho_\gamma(\mu)\,d\mu = \text{Const} \int (1-\beta\mu)^2\, d\mu$$

$$\Rightarrow \quad \rho_\gamma = \text{Const} \int (1-\beta\mu)^2\, d\mu$$

The following scenarios are:

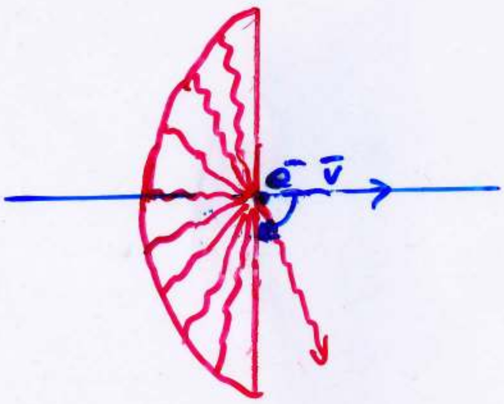
i) Photon field mainly from front:



$\psi = [\frac{\pi}{2}, \pi]$

$\mu = [0, -1]$

ii) Photon field mainly from back:



$\psi = [0, \frac{\pi}{2}]$

$\mu = [1, 0]$

Now we can compare γ-ray production rates from these two photon fields.

Then

$$\frac{P_\gamma[\text{up}]}{P_\gamma[\text{down}]} = \frac{2\int_{-1}^{0}(1-\beta\mu)^2\,d\mu}{2\int_{0}^{1}(1-\beta\mu)^2\,d\mu}$$

$$= \frac{\int_{-1}^{0}(1-2\beta\mu+\beta^2\mu^2)\,d\mu}{\int_{0}^{1}(1-2\beta\mu+\beta^2\mu^2)\,d\mu}$$

$$= \frac{\left[\mu-2\beta\frac{\mu^2}{2}+\beta^2\frac{\mu^3}{3}\right]_{-1}^{0}}{\left[\mu-2\beta\frac{\mu^2}{2}+\beta^2\frac{\mu^3}{3}\right]_{0}^{1}}$$

$$= \frac{[0]-[(-1)-\beta(-1)^2+\beta^2/3\,(-1)^3]}{[(1)-\beta(1)^2+\beta^2/3\,(1)^3]-[0]}$$

$$= \frac{1+\beta+\beta^2/3}{1-\beta+\beta^2/3}$$

$$= \frac{1+1+\frac{(1)^2}{3}}{1-1+(1)^2/3} \qquad \beta\to 1$$

$$= \frac{2+\frac{1}{3}}{\left(\frac{1}{3}\right)}$$

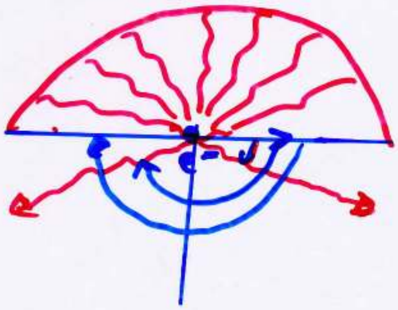
$$= \frac{\frac{6}{3}+\frac{1}{3}}{\frac{1}{3}}$$

$$= \frac{(7/3)}{(1/3)}$$

$$= 7$$

[The scattering rate with the up-stream photons is 7-times higher than with the photons coming from the back]

• Photon field from the side



$\psi = [0, \pi]$

$\mu = [1, -1]$

Compare :

$$\frac{P_\gamma[\text{up}]}{P_\gamma[\text{side}]} = \frac{2\int_{-1}^{0}(1-\beta\mu)^2\,d\mu}{\int_{-1}^{+1}(1-\beta\mu)^2\,d\mu}$$

$$= \frac{2\int_{-1}^{0}(1-2\beta\mu+\beta^2\mu^2)\,d\mu}{\int_{-1}^{+1}(1-2\beta\mu+\beta^2\mu^2)\,d\mu}$$

$$= \frac{2\left[\mu - 2\beta\frac{\mu^2}{2} + \beta^2\frac{\mu^3}{3}\right]_{-1}^{0}}{\left[\mu - 2\beta\frac{\mu^2}{2} + \beta^2\frac{\mu^3}{3}\right]_{-1}^{+1}}$$

$$= \frac{2\left(1+\beta+\frac{\beta^2}{3}\right)}{\left(1-\beta+\frac{\beta^2}{3}\right) - \left((-1) - \beta(-1)^2 + \frac{\beta^2}{3}(-1)^3\right)}$$

$$= \frac{2\left(1+\beta+\frac{\beta^2}{3}\right)}{2 - \beta + \beta + \frac{\beta^2}{3} + \frac{\beta^2}{3}}$$

$$= \frac{2\left(1+\beta+\frac{\beta^2}{3}\right)}{2+\frac{2\beta^2}{3}}$$

$$= \frac{2\left(1+1+\frac{1}{3}\right)}{\left(2+\frac{2}{3}\right)} = \frac{2\times\frac{7}{3}}{\left(\frac{8}{3}\right)} = \frac{\left(\frac{14}{3}\right)}{\left(\frac{8}{3}\right)} = \frac{7}{4}$$

Cooling Time and Compactness

The cooling time for IC emission is

$$t_{IC} = \frac{E}{P_c(\gamma)}$$

$$= \frac{\gamma m_e c^2}{\frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_{ph}}$$

$$= \frac{3 m_e c^2}{4 \sigma_T c \gamma \beta^2 U_{ph}}$$

$$\simeq \frac{3 m_e c^2}{4 \sigma_T c \gamma U_{ph}} \qquad \beta \to 1$$

This eqn offers the opportunity to introduce an important quantity, namely the compactness parameter.

Compactness Parameter:

This is essentially the ratio of the Luminosity (L) over the radius (R) of the source. The luminosity is related to the photon energy density U_{ph} in the following way:

$$L = F \times 4\pi R^2$$

$$F = \frac{L}{4\pi R^2}$$

But:

$$F = U_{ph} \times c$$

$$U_{ph} \times c = \frac{L}{4\pi R^2}$$

$$U_{ph} = \frac{L}{4\pi R^2 c}$$

$$u(\Omega) = \frac{I}{c}$$

$$u = \frac{I \Delta\Omega}{c}$$

$$= \frac{F}{c}$$

This expression for the energy density is valid outside the source. In this case the denominator is simply

$$4\pi R^2 c = \underbrace{4\pi R^2 l_{ph}/\Delta t}$$

$\llcorner$ The volume of a shell crossed by the source radiation in one second.

$$U_{ph} = \frac{L}{4\pi R^2 l_{ph}/\Delta t}$$

$$= \frac{L\Delta t}{4\pi R^2 l_{ph}}$$

$$\Rightarrow \boxed{\frac{U_{ph}}{\Delta t} = \frac{L}{4\pi R^2 l_{ph}}}$$

However we showed earlier that inside a homogeneous source a better way to determine the energy density is to determine the average time needed for a photon to cross the source. We showed earlier that for an optical thin source

$$t_{esc} = \frac{3}{4}\frac{R}{c}$$

$$\leq R/c$$

This implies that the photons that do manage to escape are not born at the centre. Hence from the formalism above

$$U_{ph} = \left(\frac{L}{V}\right) \times \Delta t_{esc}$$

$$= \frac{L}{\frac{4\pi}{3} R^2} \quad \frac{3}{4}\frac{R}{c}$$

$$= \frac{9L}{16\pi R^2 c}$$

$$= \frac{9}{4} \underbrace{\frac{L}{4\pi R^2 c}}$$

Energy density when photons can escape.

However, let's use the conventional expression for the photon energy density and substitute this into the expression for the cooling time

$$t_{IC} = \frac{3 m_e c^2}{4 \sigma_T c \gamma \; U_{ph}}$$

$$= \frac{3 m_e c^2}{4 \sigma_T c \gamma \left[\frac{L}{4\pi R^2 c}\right]}$$

$$= \frac{3\pi \, m_e c^2 \; R^2}{\sigma_T \gamma L}$$

$$= \frac{3\pi \, m_e c^3 \left(\frac{R}{c}\right) R}{\sigma_T \gamma L}$$

$$\Rightarrow \frac{t_{IC}}{\left(\frac{R}{c}\right)} = \frac{3\pi \, m_e c^3 \, R}{\sigma_T \gamma L}$$

$$= \frac{3\pi}{\gamma} \frac{1}{\ell}$$

$$\ell = \frac{\sigma_T L}{m_e c^3 R} \qquad \text{[Compactness parameter]}$$

$$\ell \propto \frac{L}{R}$$

For $\ell > 1$

$$\frac{t_{IC}}{\frac{R}{c}} < 1 \Rightarrow \text{[Compton cooling dominant]}$$

Single Particle Spectrum

We will give the main ideas behind the fact that the energy (frequency) of the scattered photons are $\sim \gamma^2$ times that of the background photons. Here are a few steps to consider

- Assume the relativistic electron travels in a region where the background photon energy density is U_{ph}. We assume the photons are monochromatic. Therefore the background photons have dimensionless energy (frequency)

$$x = \frac{h\nu}{m_e c^2}$$

- In the frame where the electron is in rest half of the photons appear to come from the front within

$$\begin{aligned} \theta' &= \frac{1}{\Gamma} \\ &= \frac{1}{\gamma} \qquad (\gamma = \Gamma) \end{aligned}$$

- The typical frequency of these photons in the electron rest frame can be determined from the Doppler eqn

$$x' = \gamma x (1 - \beta \cos\psi)$$

$$\begin{aligned} \langle x' \rangle &= \gamma \langle x \rangle \langle 1 - \beta \cos\psi \rangle \\ &= \gamma \langle x \rangle [1 - \beta \langle \cos\psi \rangle] \\ &\approx \gamma \langle x \rangle \qquad \langle \cos\psi \rangle = 0 \\ &= \gamma x \end{aligned}$$

[For monochromatic photons]

Exercise: Convince yourself that from

$$\Rightarrow x' = \gamma x (1 - \cos\psi)$$

using : $\cos\psi = \dfrac{\beta + \cos\psi'}{1 + \beta\cos\psi'}$

with $\psi' = \pi - \theta'$ $\qquad \boxed{\sin\theta' = \frac{1}{\gamma} \,;\, \cos\theta' = \beta}$

$$x' = \gamma x$$

- If we are in Thomson regime (elastic scatter in e^- rest frame) then

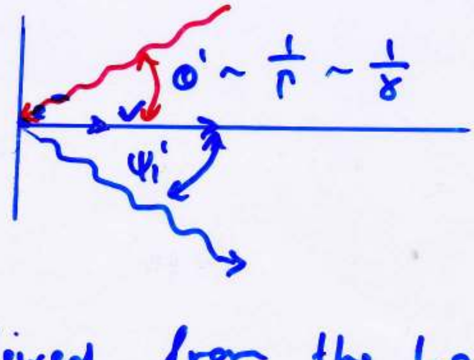
$$x' = \frac{h\nu'}{m_e c^2} < 1$$

Then

$$x_1' = x' \qquad \text{(Elastic scatter)}$$

The angular pattern of the scatter follows that of the Thomson cross section - peanut shape spatial distribution.

- All the photons in K' that were scattered within $\psi_1' \leq \frac{\pi}{2}$



now have angles viewed from the Lab system

$$\boxed{\sin\psi_1 \leq \frac{1}{\gamma}}$$

- The Lab energy of this scattered photon is then from the Doppler eqn

$$\begin{aligned} x_1 &= \delta_D \, x_1' \\ &= \frac{x_1'}{\gamma\left[1-\beta \cos\psi_1'\right]} \\ &= \frac{x_1'}{\gamma(1-\beta^2)} \\ &= \gamma \, x_1' \end{aligned}$$

$$\boxed{\psi_1 \to \frac{1}{\gamma} \; ; \; \cos\psi_1 = \beta}$$

- Then

elastic scatter

$$\left[x_1' = x' \ \text{in } K'\right]$$

$$\begin{aligned} \langle x_1 \rangle &= \gamma \langle x_1' \rangle \\ &= \gamma \left[\langle x' \rangle\right] \\ &= \gamma^2 x \end{aligned}$$

[For monochromatic photon field]

So one can see IC scattering pump-up the soft background photon field by a factor γ^2

$$\langle x_1 \rangle \propto \gamma^2 \, x$$

So that $\gamma \sim 10^4 - 10^6$ electrons can produce high energy γ-rays due to the upscattering of soft photons in the Thomson limit.

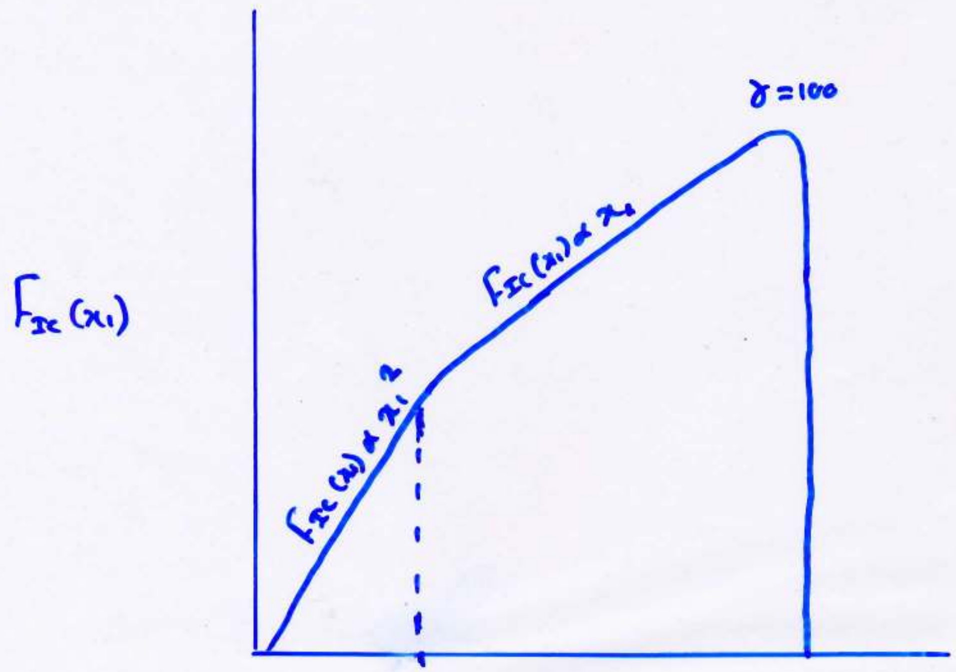
A detailed analysis (R & L 79) give the IC emissivity

$$\varepsilon_{IC}(x_1) = \frac{\sigma_T\, n\, I_0\, (1+\beta)}{4\gamma^2\beta^2 x_0}\, F_{IC}(x_1)$$

$$F_{IC}(x_1) = \frac{x_1}{x_0}\left(\frac{x_1}{x_0} - \frac{1}{(1+\beta)^2\gamma^2}\right) \quad \left\{\frac{1}{(1+\beta)^2\gamma^2} \le \frac{x_1}{x_0} \le 1\right\}$$

$$F_{IC}(x_1) = \frac{x_1}{x_0}\left[1 - \frac{x_1}{x_0}\frac{1}{(1+\beta)^2\gamma^2}\right] \quad \left\{1 \le \frac{x_1}{x_0} \le (1+\beta)^2\gamma^2\right\}$$

See Fig 5.8 for spectral behaviour (Ghisellini p 91)



$x_1/x_0 < 1$: scattered photons have less energy than incoming ones

$x_1/x > 1$: scattered photons have more energy than incoming ones

Emission from many Electrons

We have seen that the spectrum of a single e^- is peaked and that the energy of a typical photon is boosted by a factor γ^2. This is the same as the Synchrotron case. Therefore we can derive the IC emissivity in the same way as for Synchrotron radiation

Assume that the relativistic electron spectrum is a power-law

$$N(\gamma) = K\,\gamma^{-p} \qquad [\mathrm{cm}^{-3}\,\gamma^{-1}]$$

We showed earlier that

$$\underbrace{N(\gamma)\,d\gamma}_{\text{Number density of electrons within } [\gamma,\,\gamma+d\gamma]} = \underbrace{N(\epsilon)}_{\text{number density of } e^-\text{'s within } [\epsilon,\,\epsilon+d\epsilon]}\,d\epsilon$$

$$\boxed{\begin{aligned} N(\gamma) &= \frac{dN}{d\gamma} & \qquad N(\epsilon) &= \frac{dN}{d\epsilon} \\ &= K_1\,\gamma^{-p} & &= K_2\,\epsilon^{-p} \end{aligned}}$$

We assume the e^- distribution is isotropic. For simplicity we assume that the seed photon distribution is isotropic and monochromatic with frequency ν_0. Since there is a strong link between the scattered photon energy and the incoming one

$$\nu_{IC} = \frac{4}{3}\,\gamma^2\,\nu_0$$

$$\gamma^2 = \frac{3}{4}\left(\frac{\nu_{IC}}{\nu_0}\right)$$

$$\gamma = \left[\frac{3}{4}\,\frac{\nu_{IC}}{\nu_0}\right]^{1/2}$$

Then

$$\left|\frac{d\gamma}{d\nu_{IC}}\right| = \frac{1}{2}\left(\frac{3}{4}\frac{\nu_{IC}}{\nu_0}\right)^{-1/2}\left(\frac{3}{4\nu_0}\right)$$
$$= \frac{\nu_{IC}^{-1/2}}{2}\left(\frac{3}{4\nu_0}\right)^{-1/2}\left(\frac{3}{4\nu_0}\right)$$
$$= \frac{\nu_{IC}^{-1/2}}{2}\left(\frac{3}{4\nu_0}\right)^{1/2}$$

The power lost by electrons with energy between $(\gamma, \gamma+d\gamma)$ goes into radiation with frequency between $(\nu, \nu+d\nu)$. In other words an electron with energy between $[\gamma, \gamma+d\gamma]m_0c^2$ produces radiation with energy between $h(\nu, \nu+d\nu)$.

The emissivity can be written as follows:

$$\underset{}{\varepsilon_{IC}(\nu_{IC})\, d\nu_{IC}} = \underset{(sr^{-1})}{\frac{1}{4\pi}} \times \underset{(cm^{-3}\gamma^{-1})}{N(\gamma)}\ \underset{(erg\, s^{-1})}{P_{IC}(\gamma)}\ \underset{(\gamma)}{d\gamma} \quad [erg\, cm^{-3} s^{-1} sr^{-1}]$$

$$\Rightarrow \varepsilon_{IC}(\nu_{IC}) = \frac{1}{4\pi} \times N(\gamma)\, P_{IC}(\gamma)\left(\frac{d\gamma}{d\nu_{IC}}\right) [erg\, cm^{-3} s^{-1} Hz^{-1} sr^{-1}]$$
$$= \frac{1}{4\pi} K\gamma^{-p}\left[\frac{4}{3}\sigma_T c\gamma^2\beta^2 U_{ph}\right]\left[\frac{\nu_{IC}^{-1/2}}{2}\left(\frac{3}{4\nu_0}\right)^{1/2}\right]$$

$$\gamma = \left(\frac{3}{4}\frac{\nu_{IC}}{\nu_0}\right)^{1/2}$$

$$\Rightarrow \varepsilon_{IC}(\nu_{IC}) = \frac{1}{4\pi} K\left(\frac{3}{4}\frac{\nu_{IC}}{\nu_0}\right)^{-\frac{p}{2}}\left[\frac{4}{3}\sigma_T c\gamma^2\beta^2 U_{ph}\right]\left(\frac{3}{4\nu_0}\right)^{1/2}\frac{\nu_{IC}^{-1/2}}{2}$$
$$= \frac{1}{4\pi} K\left(\frac{3}{4}\frac{\nu_{IC}}{\nu_0}\right)^{-p/2}\left[\frac{4}{3}\sigma_T c\beta^2 U_{ph}\right]\left[\frac{3}{4}\frac{\nu_{IC}}{\nu_0}\right]\left[\frac{3}{4\nu_0}\right]^{1/2}\frac{\nu_{IC}^{-1/2}}{2}$$
$$= \frac{K}{4\pi}\frac{\left(\frac{4}{3}\right)^{\frac{p-1}{2}}}{2}\sigma_T c\beta^2\frac{U_{ph}}{\nu_0}\left(\frac{\nu_{IC}}{\nu_0}\right)^{-\alpha}$$

↑ power-law spectrum

$$\Rightarrow \alpha = \frac{p-1}{2}$$

We can express this as a function of photon energy

$$\mathcal{E}_{IC}(\nu_{IC}) \propto \left(\frac{\nu_{IC}}{\nu_0}\right)^{-\alpha} \quad (\mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}\ sr^{-1}})$$

$$\Rightarrow \frac{dE}{dV\,dt\,d\Omega\,d\nu} = \mathcal{E}_{IC}(\nu_{IC})$$

$$\Rightarrow \frac{dE}{dV\,dt\,d\Omega\,d(h\nu)} = \frac{\mathcal{E}_{IC}(\nu_{IC})}{h}$$

$$\mathcal{E}_{IC}(h\nu) = \frac{\mathcal{E}_{IC}(\nu_{IC})}{h}$$

$$= \frac{1}{h} \times \frac{R}{R} \times \left[\frac{K}{4\pi} \frac{\left(\frac{4}{3}\right)^{\frac{p-1}{2}}}{2} \sigma_T c \beta^2 \frac{U_{ph}}{\nu_0} \left(\frac{\nu_{IC}}{\nu_0}\right)^{-\alpha}\right]$$

$$= \frac{1}{4\pi} \frac{\left(\frac{4}{3}\right)^{\alpha}}{2} (K \sigma_T R) \beta^2 \frac{U_{ph}}{h\nu_0} \frac{1}{(R/c)} \left(\frac{\nu_{IC}}{\nu_0}\right)^{-\alpha}$$

$$= \frac{1}{4\pi} \frac{\left(\frac{4}{3}\right)^{\alpha}}{2} \frac{\tau_{IC}}{R/c} \beta^2 \frac{U_{ph}}{h\nu_0} \left(\frac{\nu_{IC}}{\nu_0}\right)^{-\alpha}$$

$\beta^2 \sim 1$ for relativistic case.

Notice that

$$\tau_{IC} = K \sigma_T R$$

Notice that for $\tau_{IC} < 1$ the optical depth τ_{IC} represents the fraction of the seed photon density

$$\frac{U_{ph}}{h\nu_0} = \frac{n\,h\nu_0}{h\nu_0} = n \ (\mathrm{cm^{-3}})$$

undergoing IC scattering in a time R/c

with $\left(\frac{\nu_{IC}}{\nu_0}\right) \sim \gamma^2$ the average gain in energy per scatter

Non-Monochromatic Seed Photons

For non-monochromatic seed photons we can adopt the previous equation as follows:

$$U_{ph} \longrightarrow U_{ph}(\nu)$$
$$[\mathrm{erg\ cm^{-3}}] \qquad [\mathrm{erg\ cm^{-3}\ Hz^{-1}}]$$
$$\nu_0 \longrightarrow \nu$$

$$d\varepsilon_{IC}(\nu_{IC}) = \frac{1}{4\pi}\,\frac{\left(\frac{4}{3}\right)^{\alpha}}{2}\,\frac{\tau_{IC}}{R/c}\,\nu_{IC}^{-\alpha}\,\frac{U_{ph}(\nu)}{\nu}\,\nu^{\alpha}\,d\nu$$

Notice that in the monochromatic case we had:

$$\varepsilon_{IC}(\nu_{IC}) = \frac{K}{4\pi}\,\frac{\left(\frac{4}{3}\right)^{\alpha}}{2}\,\sigma_T c\,\beta^{\alpha}\,\frac{U_{ph}}{\nu_0}\left(\frac{\nu_{IC}}{\nu_0}\right)^{-\alpha}$$

$$= \frac{1}{4\pi}\,\frac{\left(\frac{4}{3}\right)^{\alpha}}{2}\,\frac{\tau_{IC}}{R/c}\,\underset{\uparrow\,\beta^{\alpha}\to 1}{\beta^{\alpha}}\,\nu_{IC}^{-\alpha}\,\frac{U_{ph}}{\nu_0}\,\nu_0^{\alpha}$$

$$= \frac{1}{4\pi}\,\frac{\left(\frac{4}{3}\right)^{\alpha}}{2}\,\frac{\tau_{IC}}{R/c}\,\nu_{IC}^{-\alpha}\,\frac{U_{ph}}{\nu_0}\,\nu_0^{\alpha}$$

Therefore, for non-monochromatic seed photons

$$\varepsilon_{IC}(\nu_{IC}) = \frac{1}{4\pi}\,\frac{\left(\frac{4}{3}\right)^{\alpha}}{2}\,\frac{\tau_{IC}}{R/c}\,\nu_{IC}^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{U_{ph}(\nu)}{\nu}\,\nu^{\alpha}\,d\nu$$

Note that if the seed photons are produced by the synchrotron process (Synchrotron Self Compton (SSC))

$$\boxed{U_{ph}(\nu) \propto \nu^{-\alpha}}$$

we can solve the integral in the following way.

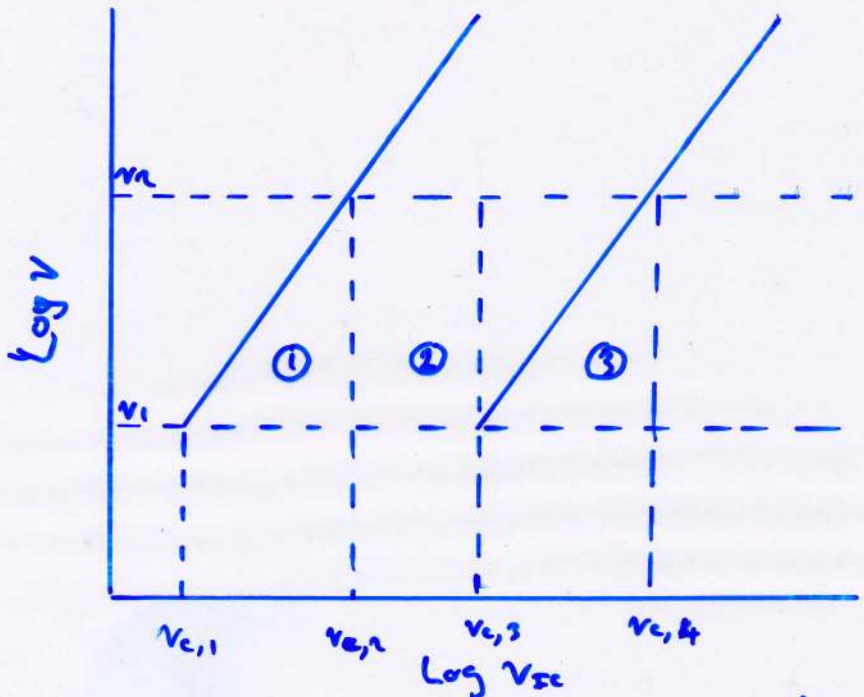
Then the integral is

$$\varepsilon_{IC}(\nu_{IC}) \propto \frac{1}{4\pi}\frac{\left(\frac{4}{3}\right)^{\alpha}}{2}\frac{P_{IC}}{R/c}\,\nu_{IC}^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{\nu^{-\alpha}}{\nu}\,\nu^{\alpha}\,d\nu$$

$$\propto \nu_{IC}^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{d\nu}{\nu}$$

$$\propto \nu_{IC}^{-\alpha}\ln\left(\frac{\nu_{max}}{\nu_{min}}\right)$$

Fig 5.9 helps to understand what is the correct ν_{min} and ν_{max} to use



On the x-axis we have the scattered frequencies ranging from

$$\nu_{c,1} = \frac{4}{3}\gamma_{min}^{2}\,\nu_1$$

$$\nu_{c,4} = \frac{4}{3}\gamma_{max}^{2}\,\nu_2$$

Then

$$\left.\begin{aligned} \nu_{c,1} &= \frac{4}{3}\gamma_{min}^2 \nu_1 \\ \nu_{c,2} &= \frac{4}{3}\gamma_{min}^2 \nu_2 \end{aligned}\right\} \quad \nu = \frac{3}{4\gamma_{min}^2}\nu_c$$

Also

$$\left.\begin{aligned} \nu_{c,3} &= \frac{4}{3}\gamma_{max}^2 \nu_1 \\ \nu_{c,4} &= \frac{4}{3}\gamma_{max}^2 \nu_2 \end{aligned}\right\} \quad \nu = \frac{3}{4\gamma_{max}^2}\nu_c$$

These eqn's tells us on each line the appropriate ν that can give ν_{IC} for specific Lorentz factor γ.

<u>There are three zones:</u>

Zone 1: Between $\nu_{c,1}$ and $\nu_{c,2} = \frac{4}{3}\gamma_{min}^2 \nu_2$

$$\nu_{min} = \nu_1$$
$$\nu_{max} = \frac{3}{4\gamma_{min}^2}\nu_{IC}$$

Zone 2: Between $\nu_{c,2}$ and $\nu_{c,3} = \frac{4}{3}\gamma_{max}^2 \nu_1$

$$\nu_{min} = \nu_1$$
$$\nu_{max} = \nu_2$$

Zone 3: Between $\nu_{c,3}$ and $\nu_{c,4} = \frac{4}{3}\gamma_{max}^2 \nu_2$

$$\nu_{min} = \frac{3}{4\gamma_{max}^2}\nu_c$$

$$\nu_{max} = \nu_2$$

We see that its only in zone 2 $[\nu_{min} = \nu_1 ; \nu_{max} = \nu_2]$ that the limits in integration coincide with the extension in frequency of the seed photon field distribution and are therefore constant. Therefore

$$\varepsilon_{IC}(\nu_{IC}) \propto \nu_{IC}^{-\alpha} \quad \text{between } [\nu_{min} = \nu_1 ; \nu_{max} = \nu_2]$$

<u>What does this mean?</u>

If IC scattering occurs in the Thomson limit between frequencies $\nu_{min} = \nu_1$ and $\nu_{max} = \nu_2$ of the background photon field, to see a well defined powerlaw spectrum the following condition must be satisfied:

The power-law spectrum of the electrons must be such that

$$N(\gamma) \propto \gamma^{-p} \quad [\gamma = \gamma_{min} ; \gamma = \gamma_{max}]$$
$$\gamma_{max} \gg \gamma_{min}$$

Remember that the low and high energy electrons scatter all the photons. Therefore

$$\Delta \varepsilon_{IC}(min) \simeq \gamma_{min}^{2} [\nu_{min} ; \nu_{max}]$$

$$\Delta \varepsilon_{IC}(max) \simeq \gamma_{max}^{2} [\nu_{min} ; \nu_{max}]$$

A power-law spectrum will be observed if

$$\boxed{\gamma_{max}^2 \nu_{min} \gg \gamma_{min}^2 \nu_{max}}$$

This is what Fig 5.9 implies

$$\left(\frac{\gamma_{max}}{\gamma_{min}}\right) \gg \left(\frac{\nu_{max}}{\nu_{min}}\right)^{1/2}$$