

COSMIC ACCELERATORS

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Prepared for the IAC2008

March 2008

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Chapter 1

Energetic particles in astrophysics

1.1 Introduction

Explosive and other very energetic events in the Universe produce, apart from copious amounts of radiation, energetic non-thermal particles. Here ‘non-thermal’ refers to the fact that these particles are distributed in energy in a manner quite different than what one encounters in *equilibrium statistical mechanics*: rather than forming a Maxwellian distribution where the number of particles scales with energy $\mathcal{E}$ as $dN/d\mathcal{E} \propto \mathcal{E}^2 \exp(-\mathcal{E}/k_b T)$ where T is the temperature and k_b is Boltzmann’s constant, the distribution one infers often takes the form of a power-law in energy:

$$\frac{dN}{d\mathcal{E}} \equiv \mathcal{N}(\mathcal{E}) = \kappa \mathcal{E}^{-s} . \quad (1.1.1)$$

The slope s of power law (1.1.1) typically lies in the range $2 \leq s \leq 3$.

We will see below that such a power-law distribution in energy essentially signals a production process without an *intrinsic* energy scale. In contrast: the Maxwellian distribution has a well-defined intrinsic energy scale: the typical thermal energy $\mathcal{E}_{\text{th}} \simeq k_b T$ per particle.

Table 1 below gives astrophysically relevant examples of energetic particles, and the way they are (or can be) detected. I will use the term particle loosely, including for instance photons such as TeV Gamma Rays. Cosmic Ray astronomy is the oldest example, and started essentially around in the 1930’s after Hess proved in 1918 that part of ionizing radiation on Earth has an extraterrestrial origin. Gamma Ray astronomy started in the 1970’s with the first dedicated Gamma-Ray satellites, but has since become a serious enterprise with the advances in detection technology in the 1990’s, and with the possibility of observing TeV Gamma Rays from the ground using specially designed telescopes such as the HESS telescope array in Namibia.

Table 1: Energetic particles in astrophysics

Particle Type	Possible Production Sites	Detection Method	Typical Energy
Galactic Cosmic Ray Nuclei	Likely: Supernova Remnants Possibly: Superbubbles (merged SNRs in OB associations) Possibly: Powerful stellar winds (Wolf-Rayet Stars/O-Stars)	Direct: Satellites (mass spectrometers, scintillators) Indirect: Atmospheric Airshowers Cherenkov/Fluorescence light from Atm. Airshowers Geo-synchrotron radio emission from Airshowers	$1 - 10^7$ GeV
Relativistic Electrons	<i>Galactic:</i> Supernova Remnants, Pulsar Winds, Accreting Compact Objects: Neutron Stars & Black Holes <i>Extragalactic:</i> Active Galactic Nuclei, Jets	Indirect: Synchrotron Radiation Inverse Compton Radiation Relativistic Bremsstrahlung	1 GeV - 1 TeV
Relativistic protons/nuclei	<i>Galactic:</i> Supernova Remnants, Accreting Compact Objects <i>Extragalactic:</i> Active Galaxies, Gamma Ray Bursts (?)	Indirect: Gamma Rays from decaying pions produced in <i>hadronic interactions</i>	$\gtrsim 100$ GeV
UHE Cosmic Rays (protons?)	<i>Extragalactic:</i> Gamma Ray Bursts and/or Active Galaxies	Indirect: Atmospheric Airshowers Atmospheric Cherenkov/Fluorescence Light Geo-synchrotron emission	$\geq 10^{18}$ eV
TeV Gamma Rays	<i>Galactic:</i> Supernova Remnants, Pulsar Winds, Cosmic Ray Interactions inside Molecular Clouds <i>Extragalactic:</i> Active Galaxies, Gamma Ray Bursts <i>Exotic:</i> Annihilation of supermassive Dark Matter particles	Indirect: Atmospheric Airshowers Atmospheric Cherenkov Light	$\gtrsim 1$ TeV
Cosmic neutrinos (not yet detected)	$\sim$ MeV Energies: Core collapse Massive Stars $>$ TeV Energies: hadronic processes in Supernova Remnants, Active galaxies (Jets) & Gamma Ray Bursts	Indirect: Weak Interactions with nuclei; Cherenkov light from reaction products Indirect: Cherenkov light from Muons from ν_μ neutrinos in km ³ water/ice detectors (ICECube, Antares, KM3Net)	$\sim 1 - 10$ MeV > 1 TeV

High-energy neutrino astrophysics is essentially being developed as we speak with the construction of the IceCube detector at the South Pole and the Antares detector in the deep Mediterranean Sea.

Of all these energetic astro-particles, only the Galactic Cosmic Rays can in principle be measured directly using a mass-spectrometer or a scintillator on a satellite platform in Earth orbit. In all the other cases the detection is indirect: relativistic electrons are 'visible' through the electromagnetic radiation they produce: *synchrotron radiation* when they spiral around a magnetic field, *inverse Compton radiation* when they collide with low-energy photons and boost them to higher energy, and *Bremsstrahlung* (also called *free-free radiation*) when they are deflected in electric fields of ions or other (low-energy/thermal) electrons. All these processes produce *continuum radiation*, as opposed to the *line radiation* that is produced by electronic transitions in atoms. Relativistic nuclei undergo the same processes in principle, but the intensity of the electromagnetic radiation from protons or other ions is heavily suppressed. This can be seen as follows: the radiation loss formula for an accelerated charge reads:

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{rad}} = -\frac{2}{3} \frac{q^2 \gamma^6}{c} \left(|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}|^2 \right) \quad (1.1.2)$$

Here q is the charge of the particle, $\boldsymbol{\beta} = \mathbf{v}/c$ with $\mathbf{v}$ the particle velocity, $\dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt$ and $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor. Let us consider ultrarelativistic particles with $\beta = |\boldsymbol{\beta}| \simeq 1$ and with $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$ so that the particle is mostly deflected. In that case Eqn. (1.1.2) reduces to

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{rad}} = -\frac{2}{3} \frac{q^2 \gamma^4}{c} |\dot{\boldsymbol{\beta}}|^2. \quad (1.1.3)$$

If the mass of the particle is m the typical acceleration due to a force $F_{\perp}$ perpendicular to the direction of motion corresponds to:

$$|\dot{\boldsymbol{\beta}}| \sim \frac{F_{\perp}}{\gamma m c}. \quad (1.1.4)$$

The radiated power of a particle with energy $\mathcal{E} = \gamma m c^2$ then is of order

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\text{rad}} \simeq -\frac{2}{3} \frac{q^2 \gamma^2 F_{\perp}^2}{m^2 c^3} = -\frac{2}{3} \frac{q^2 \mathcal{E}^2 F_{\perp}^2}{m^4 c^7}. \quad (1.1.5)$$

This shows that when relativistic electrons and protons of the same energy $\mathcal{E}$ are subjected to the same force, the total amount of radiation from the protons is a factor $(m_p/m_e)^4 \sim 10^{13}$ less intense than the radiation from the electrons.

Protons (and nuclei) in distant sources can only be detected through some intermediary process that produces a photon or, in the case of neutrino astronomy, energetic neutrino's. Photons can be produced by energetic protons through a hadron-hadron collision or a hadron-photon collision of the form

$$p + N \implies p + N + \pi^0 \text{ or } p + \gamma \implies p + \pi^0, \quad (1.1.6)$$

followed by the decay of the neutral pions produced in these collisions:

$$\pi^0 \implies 2\gamma. \quad (1.1.7)$$

Here N stands for a nucleon (a proton or a neutron that may or may not reside inside a nucleus), γ stands for a photon and π^0 is a neutral pion. Typically, the gamma rays carry away 10% of the proton energy. Observationally, the problem is to distinguish these 'hadronic' Gamma Rays from 'leptonic' Gamma Rays that are produced by electrons, for instance by the process of inverse-Compton scattering. The only way to do that is to determine the precise spectrum of the Gamma Rays over a wide range of photon energies and -at the same time- determine the population of relativistic electrons in the source, and then compare these spectra with models. Usually the electron distribution as a function of electron energy inside the source can be determined by looking at the synchrotron emission from these same electrons that is visible at much lower photon energies (from radio to X-Rays). Recent observations of a few supernova remnants with the HESS Gamma Ray array has given strong indications that relativistic protons are indeed present in these SNRs with an energy of 10 – 100 TeV. This strengthens the hypothesis that the Galactic Cosmic Rays are produced inside Supernova Remnants.

There is a related set of reactions that will take place concurrently, and which are of importance for neutrino-astronomy. In these reactions the incoming proton undergoes an *isospin flip* and turns into a neutron (n), while the charge is carried away by a charged pion:

$$p + N \implies n + N + \pi^+ \text{ or } p + \gamma \implies n + \pi^+. \quad (1.1.8)$$

This is followed by the decay of the charged pions produced in these collisions:

$$\pi^+ \implies \mu^+ + \nu_\mu \quad (1.1.9)$$

This is immediately followed by the decay of the muons¹:

$$\mu^+ \implies e^+ + \nu_e + \bar{\nu}_\mu . \quad (1.1.10)$$

In total three neutrinos are produced in these reactions. Ultra-relativistic protons loose roughly half their energy in such a reaction. This energy is then divided almost equally between the final four particles so the typical energy per neutrino is about 10% of the original proton energy. The observation of an astrophysical point source of high-energy neutrino's (energies well above 1 TeV) would be a conclusive proof that hadrons as well as electrons are accelerated in astrophysical accelerators like Supernova Remnants since purely leptonic models (i.e. models without any energetic hadrons) do not produce any high-energy neutrinos.

¹Anti-neutrino's are designated ny $\bar{\nu}$.

Chapter 2

Astrophysical accelerators

2.1 Astrophysical acceleration: basic considerations

One of the first two things to realize about the acceleration of particles in the astrophysical context is the fact that all acceleration mechanisms are [1] *electromagnetic* in nature and [2] must involve material *motions* in the form of waves or shocks in a *magnetized* and ionized gas, such as the hot-phase interstellar medium. The first requirement is a simple consequence of the fact that cosmic rays are *collisionless* in the strict sense of the word, for instance: the Galactic disk with a diameter of $\sim 70,000$ light year is completely transparent to cosmic rays for Coulomb collisions with nuclei in the interstellar medium. Although collisions play a role as an energy-loss mechanism, e.g. ionization losses or pion-production losses at high energy, they are totally unimportant for the scattering of particles. As we will see below, most models still rely on some form of scattering, not by other particles but by the magnetic fields that are associated with low-frequency waves.

The second requirement is a consequence of the fact that astrophysical plasma's are highly conducting, so that any large-scale electric field is given by the well-known condition of ideal magnetohydrodynamics (MHD) that says that the electric field is induced by the bulk motion of the highly-conducting material:

$$\mathbf{E} = -\frac{\mathbf{V}}{c} \times \mathbf{B} = -\Gamma\boldsymbol{\beta} \times \mathbf{B}' . \quad (2.1.1)$$

Here $\mathbf{V} = \boldsymbol{\beta}c$ is the bulk velocity of the plasma, $\Gamma = 1/\sqrt{1-\boldsymbol{\beta}^2}$ the Lorentz-factor associated with the bulk motion, $\mathbf{B}$ is the magnetic field in the laboratory frame, and $\mathbf{B}'$ is the magnetic field in the rest frame of the material. This large-scale electric field is essential for acceleration as the magnetic component of the Lorentz force on a particle by itself does no work, and does not change the particle energy: it merely deflects a charge.

In fact, the equation of motion for a charged particle gives the change of kinetic energy $K = (\gamma - 1)mc^2$ as

$$\frac{dK}{dt} = q(\mathbf{E} \cdot \mathbf{v}) . \quad (2.1.2)$$

Here q is the particle charge and $\mathbf{v}$ the particle velocity.

The importance of bulk motion was first realized by Fermi¹, who described two simple acceleration models which have served as 'templates' for all subsequent work:

- **Stochastic (Fermi-II) acceleration** by scattering off randomly moving magnetized clouds in a turbulent medium;
- **Regular (Fermi-I) acceleration** by reflection off shocks.

In the first model particles diffuse in energy in such a way that the *mean* energy per particle increases, a simple consequence of the fact that head-on collisions which result in an energy gain are slightly more frequent than overtaking collisions in which a particle loses energy. In the second model all particles systematically gain energy. Parker² considered a simple physical realization of the Fermi-II model in which the moving clouds are replaced by MHD waves. His suggestion was followed by many stochastic acceleration models where particles diffuse in energy due to the random momentum changes induced by the electric field of low-frequency MHD waves. Such models have now gone largely out-of-fashion, and I will only briefly consider them here. The basic idea behind these two mechanisms is illustrated in the Figure below.

Intermezzo: the MHD condition

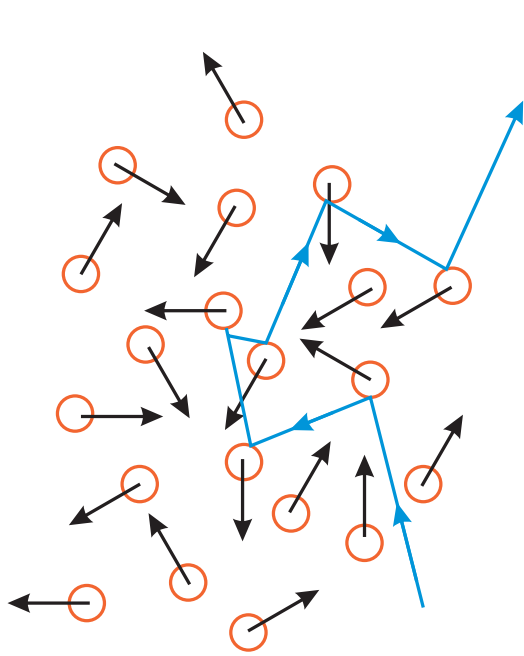
In magnetohydrodynamics (MHD) one uses a simple relation between the fluid velocity, the magnetic field and the electric field. In this section, I will derive this relationship using a simple model.

The electric field and current density can be found by a simple argument, which balances the effect of inter-particle collisions with the electromagnetic force on charged particles. Consider a non-relativistic charged particle in the ionized gas: a hydrogen plasma consisting of electrons and protons. Its equation of motion reads:

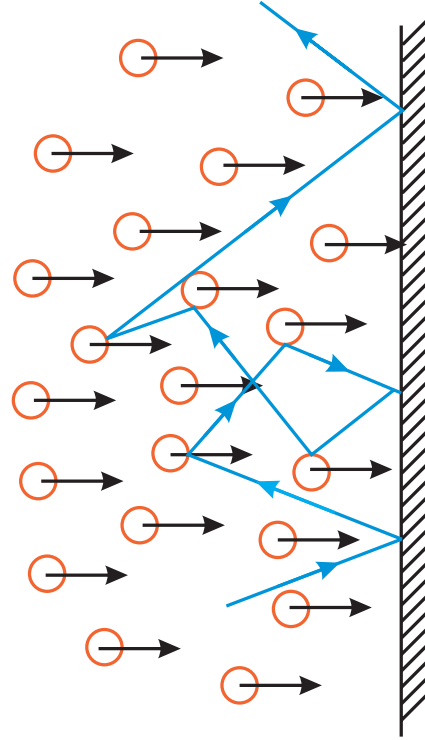
$$\frac{d\mathbf{v}}{dt} = \frac{q}{m} \left[\mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \right] - \frac{1}{\tau_s} (\mathbf{v} - \mathbf{V}) . \quad (2.1.3)$$

¹E. Fermi, *Phys. Rev.* **75**, 1169, 1949.

²E.N. Parker, *Phys. Rev.* **99**, 241, 1955



Stochastic Fermi Acceleration



Regular Fermi Acceleration

Figure 2.1: The basic idea of stochastic (left) and regular (right) Fermi acceleration. Scattering centers (red) scatter a light particle in collisions that are elastic in the rest frame of the scattering center. The trajectory of the scattered particle is shown in blue. In **stochastic Fermi acceleration** the scattering centers move in a random direction. A particle gains energy in a 'head-on' collision with a scattering center, and loses energy in an overtaking collision. However, since the head-on collisions are more frequent than the overtaking collisions, there is a net energy gain. In the astrophysical application of this idea, the scattering centers are in fact magnetized waves, where the Lorentz force associated with the fluctuating magnetic fields in the waves is responsible for the deflection (see text) of a charged particle. In **regular Fermi acceleration** there is a special geometry: the scattering centers move all in the same direction. In this figure, an elastic wall confines the particle, which follows the blue trajectory. Reflection off the wall forces the particle to return into the gas of scattering centers again and again. In each cycle between two reflection off the wall, the particle gains energy, simply because the first collision after reflection is always a head-on collision. In the astrophysical application of this idea the wall is replaced by a shock, and a second set of scattering centers behind the shock. The scattering of charged particles is again provided by the fluctuating magnetic fields of magnetized plasma waves.

Here the first term on the right-hand-side is the Lorentz force on the with charge q , and the second term is a frictional force due to inter-particle collisions which tries to equalize the velocity of all particles to the average velocity $\mathbf{V}$. The quantity τ_s is the typical time between two subsequent collisions.

For slow phenomena one neglect the left-hand-side of this equation, putting the inertial force to zero: $m (d\mathbf{v}/dt) \approx 0$. One then finds the deviation between the velocity of the individual charge species and the mean velocity:

$$\mathbf{v} - \mathbf{V} = \frac{q\tau_s}{m} \left[\mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \right] \quad (2.1.4)$$

Now consider the whole hydrogen plasma, consisting of electrons (charge $q = -e$) and hydrogen ions (protons with charge $q = e$). I will assume that the plasma is electrically neutral, so that the *net* charge density vanishes:

$$\sum_{\beta=e,i} q_{\beta} n_{\beta} = e (n_i - n_e) = 0. \quad (2.1.5)$$

Here I use the index β which takes the 'values' e (for electron) and i (for ion) to distinguish the particle species in the mixture. Using relation (2.1.4) for both the electrons and the ions, and multiplying this equation by their respective density one finds the current density of each species:

$$\begin{aligned} \mathbf{j}_{\beta} &= n_{\beta} q_{\beta} \mathbf{v}_{\beta} \\ &\approx n_{\beta} q_{\beta} \mathbf{V} + \frac{n_{\beta} q_{\beta}^2 \tau_{s\beta}}{m_{\beta}} \left[\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right]. \end{aligned} \quad (2.1.6)$$

Here I have assumed that the average velocity for both species equals $\mathbf{V}$ to lowest order so that I can replace $\mathbf{v}_{\beta}$ by $\mathbf{V}$ in the Lorentz-force term. The first term in relation (2.1.6) is an *advection current* that is associated with the mean motion of the particles. This term can always be transformed away by using a frame-of-reference where $\mathbf{V} = 0$. The second term is a *conduction current* that gives the current associated with the small difference between the velocity of each individual particle species and the average velocity which results from the balance (2.1.4) between the Lorentz force and the friction force on the particles. This conduction current is regulated by the collision rate $\nu_s = 1/\tau_s$.

Summing over the two charge species and using the neutrality condition (2.1.5) one finds the *net* current density:

$$\begin{aligned} \mathbf{j} &= \sum_{\beta=e,i} \mathbf{j}_{\beta} \\ &\approx \sum_{\beta=e,i} \frac{n_{\beta} q_{\beta}^2 \tau_{s\beta}}{m_{\beta}} \left[\mathbf{E} + \frac{1}{c} (\mathbf{V} \times \mathbf{B}) \right]. \end{aligned} \quad (2.1.7)$$

The advection current gives no net contribution, and only the conduction current remains.

Relation (2.1.7) can be used to define a scalar *electrical conductivity*, σ , by writing

$$\mathbf{j} = \sigma \left[\mathbf{E} + \frac{1}{c} (\mathbf{V} \times \mathbf{B}) \right], \quad (2.1.8)$$

where

$$\sigma = \sum_{\beta=e,i} \frac{n_{\beta} q_{\beta}^2 \tau_{s\beta}}{m_{\beta}} \approx \frac{n_e e^2 \tau_{se}}{m_e}. \quad (2.1.9)$$

Because the electron mass is much smaller than the proton mass ($m_p/m_e \approx 1836$) the electron contribution to σ usually dominates.

Consider now the limit of a large electrical conductivity: $\sigma \rightarrow \infty$, which occurs when the time between collisions is long. This is typically the case in very tenuous astrophysical plasmas. In that case one has:

$$\mathbf{E} + \frac{1}{c} (\mathbf{V} \times \mathbf{B}) = \frac{\mathbf{j}}{\sigma} \equiv \eta \mathbf{j} \rightarrow 0. \quad (2.1.10)$$

Here $\eta = 1/\sigma$ is the *electrical resistivity* of the ionized gas. The current density $\mathbf{j}$ must always remain *finite*!

Therefore, in an ideally conducting gas where $\eta \rightarrow 0$, the electric field must be related to the average velocity $\mathbf{V}$ and the magnetic field $\mathbf{B}$ by the *ideal magnetohydrodynamic condition*:

$$\mathbf{E} = -\frac{1}{c} (\mathbf{V} \times \mathbf{B}).$$

(2.1.11)

If one uses this expression for $\mathbf{E}$ in Eqn. (2.1.3) one sees that the Lorentz force vanishes for a charged particle moving with the average velocity so that $\mathbf{v} = \mathbf{V}$. Relation (2.1.11) is sometimes referred to as the condition of a *frozen-in field*. This means that the magnetic field lines are dragged along by the flow in a perfectly conducting fluid so that the magnetic flux $\Phi_M = \mathbf{B} \cdot d\mathbf{O}$ through a surface $d\mathbf{O}$ moving with the fluid is conserved.

The MHD condition only gives a relation between the magnetic field, the plasma velocity and the components of the electric field perpendicular to $\mathbf{B}$. In the direction along the magnetic field (denoted by the subscript $\parallel$) one has

$$J_{\parallel} = \sigma E_{\parallel} \quad (2.1.12)$$

If one wants to keep $J_{\parallel}$ finite for $\sigma \rightarrow \infty$ one must *also* demand $E_{\parallel} = 0$.

Even though the derivation has been for a gas of non-relativistic particles, one can show that a similar conclusion holds for a gas of relativistic particles. The reason is simple: the Lorentz force is the same for a relativistic and a non-relativistic particle:

$$\mathbf{F}_L = q \left(\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right) . \quad (2.1.13)$$

This means that the MHD condition (2.1.1) remains correct, even when $|\mathbf{V}| \sim c$. There is one complication, however: one has to link the electromagnetic fields in different frames using fully relativistic Lorentz-transformations. The Lorentz-transformation laws for the electromagnetic field, which give the electromagnetic fields $(\mathbf{E}', \mathbf{B}')$ in a frame K' moving with velocity $\mathbf{V}$ in terms of the electromagnetic fields $(\mathbf{E}, \mathbf{B})$ in the lab frame K , read³:

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} \quad , \quad \mathbf{E}'_{\perp} = \Gamma \left(\mathbf{E}_{\perp} + \frac{\mathbf{V}}{c} \times \mathbf{B} \right) ; \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \quad , \quad \mathbf{B}'_{\perp} = \Gamma \left(\mathbf{B}_{\perp} - \frac{\mathbf{V}}{c} \times \mathbf{E} \right) . \end{aligned} \quad (2.1.14)$$

Here $\Gamma = 1/\sqrt{1 - V^2/c^2}$ is the Lorentz-factor associated with the velocity between the two frames, and the two subscripts $\parallel$ and $\perp$ refer to the directions along and perpendicular to $\mathbf{V}$.

Nevertheless, to MHD condition $\mathbf{E}' = \mathbf{0}$ (the vanishing of the electric field in the rest frame of the material) still corresponds to $E_{\parallel} = 0$, and $\mathbf{E}_{\perp} = -\mathbf{V} \times \mathbf{B}/c$, just as in the non-relativistic case.

³See for instance: L.D. Landau & E.M. Lifshitz, *Classical Theory of Fields*, Ch. 24, Pergamon Press, Oxford, 1975.

2.2 Maximum attainable energy in Cosmic Accelerators

The electromagnetic character of astrophysical particle acceleration allows one to place a firm upper limit on the energy of the particles that can be produced in a source. Relations (2.1.2) and (2.1.1) together imply that the energy change $\Delta\mathcal{E}$ of a particle with charge q is of order:

$$\Delta\mathcal{E} \sim q \int d\mathbf{s} \cdot \mathbf{E} = -\frac{q}{c} \int d\mathbf{s} \cdot (\mathbf{V} \times \mathbf{B}_\perp) . \quad (2.2.1)$$

Here $d\mathbf{s} \equiv \mathbf{v} dt$ is a infinitesimal section of the particle's orbit. The magnetic field $\mathbf{B}_\perp$ that enters this expression is the field in the laboratory frame. If one combines the transformation law (2.1.14) and the MHD condition (2.1.1), it is easily seen that the laboratory magnetic field is related to the magnetic field in rest frame in the material by:

$$\mathbf{B}_\perp = \Gamma_s \mathbf{B}'_\perp , \quad (2.2.2)$$

where Γ_s is the Lorentz factor associated with the motion in the source. As an order-of-magnitude, a source with typical size R_s , and involving a velocity, such as a shock velocity, equal to $V_s = \beta_s c$ can only produce particles with $\mathcal{E} \leq \mathcal{E}_{\max}$, where

$$\mathcal{E}_{\max} \sim q\beta_s R_s B \sim q\beta_s \Gamma_s R_s B' . \quad (2.2.3)$$

This is an upper limit, not only because it assumes a favorable geometry that maximizes the integral in expression (2.2.1), but also because it neglects the effect *energy losses*, such as the synchrotron losses suffered by electrons in a magnetic field, or the pion-production losses suffered by energetic protons in an intense radiation field.

Estimate (2.2.3) is the basis of the *Hillas Diagram* (named after the British Cosmic Ray Physicist A.M. Hillas) which shows the typical energy that can be produced in an astrophysical source as a function of the size of the source and the typical magnetic field. An example is shown in the Figure on the next page.

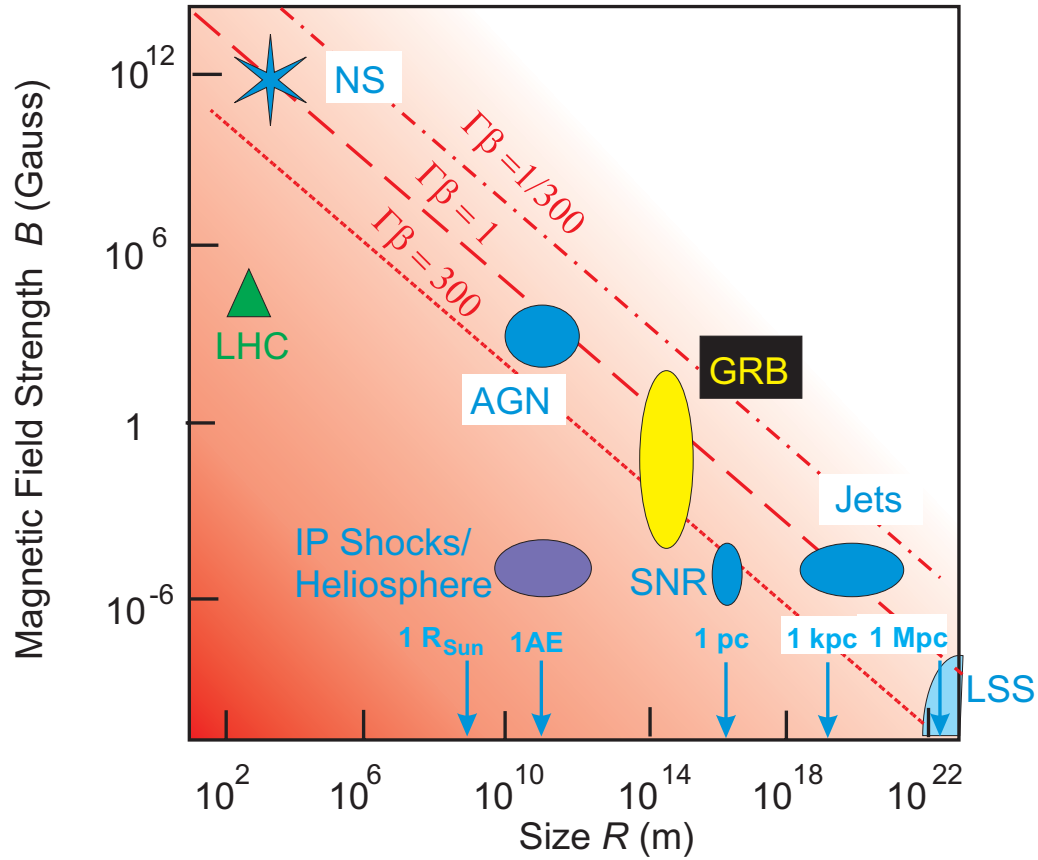


Figure 2.2: *The Hillas diagram. On the horizontal axis is the typical size of the source (in meters), and the vertical axis gives the typical magnetic field B' in Gauss. The three diagonal red lines correspond with $\mathcal{E}_{\text{max}} = 10^{20}$ eV, for three values of $\Gamma_s\beta_s$. This energy corresponds to the typical energy of the most energetic particles ever detected in Cosmic Ray experiments. The location in this diagram is shown for a number of cosmic accelerators: AGN: Active Galactic Nuclei, NS: Neutron Stars, GRB: Gamma Ray Bursts, IP Shocks: Interplanetary Shocks, Jets: the kpc-Mpc- sized jets of Active Galaxies, and LSS: the shocks associated with Large Scale Structure formation in the Universe. Also shown is the location of the Large Hadron Collider (LHC). This diagram neglects the possible effect of losses.*

2.3 The Billiard Ball analogy

The MHD condition (2.1.1) implies that the electric field in the rest-frame of an infinitely conducting plasma vanishes:

$$\mathbf{E}' = 0 . \quad (2.3.1)$$

There is only a magnetic field, and as a consequence the particle energy is conserved in the rest frame. Using accents to denote quantities in the rest frame of the plasma, one has:

$$\frac{d\mathcal{E}'}{dt'} = q (\mathbf{v}' \cdot \mathbf{E}') = 0 . \quad (2.3.2)$$

So, if a charged particle is deflected by a magnetized cloud, the energy $\mathcal{E}'$ is conserved in the rest frame of the cloud!

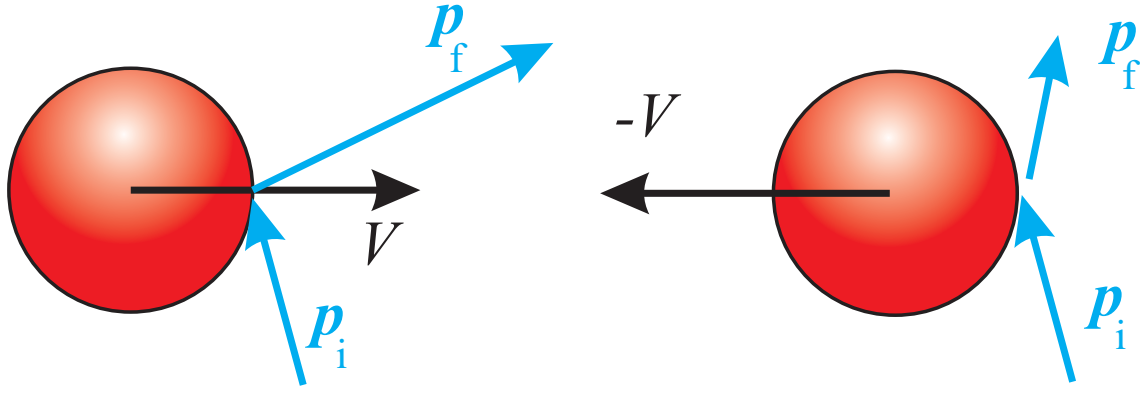
We can use this property to model the acceleration that results from the interaction between a charged particles and moving magnetic fields. This mechanical analogy is the reflection of a particle of small mas by a scattering center of large mass, say a billiard ball, see the Figure below. If we take the limit of infinite billiard ball mass, the particle will be scattered elastically, just like what happens for scattering of individual charges by magnetized clouds.

I will denote quantities in the scattering center rest frame K' by primes. Quantities measured by an observer in the laboratory frame K are unprimed. Physical quantities in the two frames are connected by the Lorentz transformations:

$$\begin{aligned} p_{\parallel} &= \Gamma \left(p'_{\parallel} + \frac{\mathcal{E}'}{c^2} \mathbf{V} \right) , \quad p_{\perp} = p'_{\perp} ; \\ \mathcal{E} &= \Gamma (\mathcal{E}' + \mathbf{p}' \cdot \mathbf{V}) . \end{aligned} \quad (2.3.3)$$

Here Γ is the Lorentz-factor associated with the relative velocity between the two frames:

$$\Gamma = 1/\sqrt{1 - \frac{V^2}{c^2}} . \quad (2.3.4)$$



Head-on collision

Overtaking collision

Figure 2.3: *The billiard-ball analogy for the acceleration of charged particles by moving magnetized disturbances (clouds, shocks or waves). In a head-on collision the particle momentum increases, $|\mathbf{p}_f| > |\mathbf{p}_i|$, in an overtaking collision the reverse is true, $|\mathbf{p}_f| < |\mathbf{p}_i|$. The momentum change can be calculated by using the fact that the collision is elastic in the rest frame of the billiard ball.*

The two subscripts $\parallel$ and $\perp$ refer to the directions along and perpendicular to $\mathbf{V}$. Note that $\mathbf{p}' \cdot \mathbf{V} = p'_\parallel V$.

Expressions (2.3.3) give the energy and momentum in K in terms of the corresponding quantities in K' . The inverse transformation simply follows by replacing $\mathbf{V}$ by $-\mathbf{V}$ and interchanging primed and unprimed quantities. This is physically obvious: seen from the point-of-view of an observer at rest in frame K' , the lab frame moves with velocity $-\mathbf{V}$.

Consider a single scattering event as described by two observers, one in the lab frame K and one in the scattering center rest frame K' , as:

$$\begin{aligned}
 &\text{in the laboratory frame } K: \quad \begin{cases} \mathbf{p}_i \implies \mathbf{p}_f \neq \mathbf{p}_i \\ \mathcal{E}_i \implies \mathcal{E}_f \neq \mathcal{E}_i \end{cases} \\
 &\text{in scattering center frame } K': \quad \begin{cases} \mathbf{p}'_i \implies \mathbf{p}'_f \neq \mathbf{p}'_i \\ \mathcal{E}'_i \implies \mathcal{E}'_f = \mathcal{E}'_i \end{cases}
 \end{aligned} \tag{2.3.5}$$

Here the subscripts i (f) denote values before (after) the scattering event.

The relativistic relation between energy and momentum,

$$\mathcal{E}' = \sqrt{(p')^2 c^2 + m^2 c^4} , \quad (2.3.6)$$

implies that the magnitude of momentum $|\mathbf{p}'|$ in the scattering center frame will not change.

One can employ the Lorentz transformations from K to K' to express scattering center frame energy and momentum before scattering in the corresponding lab frame quantities:

$$\begin{aligned} \mathcal{E}'_i &= \Gamma (\mathcal{E}_i - \mathbf{V} \cdot \mathbf{p}_i) , \\ \mathbf{p}'_{\parallel i} &= \Gamma \left(\mathbf{p}_{\parallel i} - \frac{\mathcal{E}_i}{c^2} \mathbf{V} \right) . \end{aligned} \quad (2.3.7)$$

I now assume the simple case of *specular reflection* where

$$\mathbf{p}'_{\parallel f} = -\mathbf{p}'_{\parallel i} , \quad \mathbf{p}'_{\perp f} = \mathbf{p}'_{\perp i} . \quad (2.3.8)$$

The parallel component of momentum is reversed, while the perpendicular component remains unaffected. Note that this satisfies the elasticity condition in the scattering center frame.

From the condition of elastic scattering ($\mathcal{E}'_i = \mathcal{E}'_f$) one then immediately finds the lab frame energy after scattering, simply by using the inverse Lorentz transformation from frame $K' \Rightarrow K$ together with the specular reflection condition (2.3.8):

$$\begin{aligned} \mathcal{E}_f &= \Gamma (\mathcal{E}'_f + \mathbf{V} \cdot \mathbf{p}'_f) = \Gamma (\mathcal{E}'_i - \mathbf{V} \cdot \mathbf{p}'_i) \\ &= \Gamma^2 \left[\left(1 + \frac{V^2}{c^2} \right) \mathcal{E}_i - 2 (\mathbf{V} \cdot \mathbf{p}_i) \right] . \end{aligned} \quad (2.3.9)$$

Using the general relation between momentum, energy and velocity,

$$\mathbf{p} = \gamma m \mathbf{v} = \frac{\mathcal{E}}{c^2} \mathbf{v} , \quad (2.3.10)$$

and Eqn. (2.3.7), one finds the energy change resulting from the encounter:

$$\Delta\mathcal{E} = \mathcal{E}_f - \mathcal{E}_i = 2\Gamma^2 \left(\frac{V^2}{c^2} - \frac{\mathbf{V} \cdot \mathbf{v}_i}{c^2} \right) \mathcal{E}_i. \quad (2.3.11)$$

From now on I will drop the subscript i . This result is true for arbitrary scattering center velocity, although I will often assume $V \ll v \leq c$.

One can write relation (2.3.11) in the form

$$\boxed{\frac{\Delta\mathcal{E}}{\mathcal{E}} = 2\Gamma^2 \frac{(\mathbf{V} - \mathbf{v}) \cdot \mathbf{V}}{c^2}}. \quad (2.3.12)$$

This expression allows us to distinguish two situations:

- In a *head-on collision*, where the relative velocity between particle and scattering center is such that $(\mathbf{v} - \mathbf{V}) \cdot \mathbf{V} < 0$, a particle *gains* energy in the lab frame as a result of the scattering event: $\Delta\mathcal{E} > 0$;
- In an *overtaking collision* on the other hand, with a relative velocity between particle and scattering centre such that $(\mathbf{v} - \mathbf{V}) \cdot \mathbf{V} > 0$, the particle *loses* energy in the lab frame as a result of the interaction with the scattering center: $\Delta\mathcal{E} < 0$.

The calculation presented above uses specular reflection, as it leads to a particularly simple relationship between the particle momentum before and after the scattering. However, the conclusion (Eqn. 2.3.12) is more *generally* valid as an order-of-magnitude estimate for the energy change $\Delta\mathcal{E}$. In particular, if the motion of the plasma is non-relativistic, with $|\mathbf{V}| \ll c$, and if the particle velocity is much larger than the bulk plasma velocity so that $|\mathbf{v}| \gg |\mathbf{V}|$, the typical energy change satisfies

$$\frac{|\Delta\mathcal{E}|}{\mathcal{E}} \sim \frac{vV}{c^2}. \quad (2.3.13)$$

An alternative expression considers the change Δp in the magnitude of the momentum. Quite generally one has $p = \gamma mv$, $\Delta\mathcal{E} = v \Delta p$, and $\mathcal{E} = \gamma mc^2 = c^2 p/v$. From this one finds $\Delta p/p = (c/v)^2 \Delta\mathcal{E}/\mathcal{E}$, and (2.3.13) is equivalent with:

$$\frac{|\Delta p|}{p} \sim \frac{V}{v} . \quad (2.3.14)$$

There are now two possibilities, depending how the scattering center velocity $\mathbf{V}$ is distributed (see the Figure 2.1 above). In a special geometry one can construct a situation where (in a properly chosen reference frame) only energy-increasing head-on collisions or elastic collisions occur. This situation is known as *regular Fermi acceleration* as each particle increases its energy in a systematic fashion. If both head-on and overtaking collisions occur a particle can both gain and lose energy, and a *random walk* in energy results. This is the situation that is referred to as *stochastic Fermi acceleration*. In that case, the net energy gain results from the fact that head-on collisions are a little bit more frequent than the overtaking collisions.

2.4 Regular and stochastic acceleration

Let us first consider what happens in stochastic Fermi acceleration, where the scattering centers move randomly in the sense that the scattering center velocity $\mathbf{V}$ is distributed isotropically. For simplicity I will assume that the magnitude of the velocity $V = |\mathbf{V}|$ is the same for all scattering centers. In the limit of relativistic particles ($v \simeq c$) and $V \ll c$ the energy change per collision is:

$$\Delta \mathcal{E} \simeq 2 \left(\frac{V^2}{c^2} - \frac{(\mathbf{v} \cdot \mathbf{V})}{c^2} \right) \mathcal{E} = 2 \left(\frac{V^2}{c^2} - \frac{V \cos \theta}{c} \right) \mathcal{E} . \quad (2.4.1)$$

Here θ is the angle between the velocity of the particle and the velocity of the scattering center. Head-on collisions occur for $\frac{1}{2}\pi < \theta < \pi$ while overtaking collisions occur for $0 \leq \theta \leq \frac{1}{2}\pi$.

If the density of scattering centers is n_* and their cross section (target area) is σ_* the encounter rate for given $\mathbf{v}$ and $\mathbf{V}$ is

$$\mathcal{R}_* = n_* \sigma_* |\mathbf{v} - \mathbf{V}| \quad (2.4.2)$$

For $v \gg V$ this can be approximated by:

$$\mathcal{R}_* \simeq n_* \sigma_* v \left(1 - \frac{V}{v} \cos \theta \right) . \quad (2.4.3)$$

Here I have neglected a term of order $V^2/v^2 \simeq V^2/c^2$.

From this expression it is immediately obvious that head-on collisions with $\cos \theta < 0$ are slightly more frequent than overtaking collisions that have $\cos \theta > 0$. The mean energy change per unit time can be found by averaging over all possible values of $\cos \theta$, that is: all possible encounter geometries. If the energetic particles and/or the scattering centers have an isotropic distribution of velocities the averaging is particularly simple. Defining $\mu \equiv \cos \theta$, the average $\overline{Q}$ of some quantity $Q(\theta)$ becomes

$$\overline{Q} = \frac{1}{2} \int_{-1}^{+1} d\mu Q(\mu) . \quad (2.4.4)$$

In particular we have $\overline{\mu} = 0$ and $\overline{\mu^2} = 1/3$. The mean energy gain per unit time therefore equals (putting $v = c$):

$$\begin{aligned} \frac{d\overline{\mathcal{E}}}{dt} &= n_* \sigma_* c \mathcal{E} \left[\int_{-1}^{+1} d\mu \left(1 - \frac{V\mu}{c} \right) \left(\frac{V^2}{c^2} - \frac{V\mu}{c} \right) \right] \\ &= \frac{8}{3} n_* \sigma_* c \left(\frac{V}{c} \right)^2 \mathcal{E} . \end{aligned} \quad (2.4.5)$$

The average collision rate $\overline{R}_*$ and associated mean-free-path λ (i.e. the typical distance a particle travels between collisions) are given by:

$$\overline{R}_* = n_* \sigma_* c , \quad \lambda = \frac{1}{n_* \sigma_*} . \quad (2.4.6)$$

Result (2.4.5) can be written as:

$$\frac{d\overline{\mathcal{E}}}{dt} = \frac{8c}{3\lambda} \left(\frac{V}{c} \right)^2 \mathcal{E} \equiv \alpha \mathcal{E} .$$

(2.4.7)

Here $\alpha = 8V^2/3c\lambda$ has the dimension of $[\text{time}]^{-1}$, so $\tau_{\text{acc}} = 1/\alpha$ can be thought of as a typical acceleration time: relation (2.4.7) implies that the typical energy of a particle grows as $\mathcal{E}(t) = \mathcal{E}_0 e^{\alpha t} = \mathcal{E}_0 e^{t/\tau_{\text{acc}}}$. This is the classical expression first derived by Enrico Fermi in 1949⁴.

⁴E. Fermi 1949: *Phys. Rev.* **75**, 1169.

The *mean* energy change per collision is a factor $\sim V/c$ smaller than the magnitude of the typical energy change in an individual collision. In fact, to leading order in V/c one has:

$$\frac{\overline{\Delta\mathcal{E}}}{\mathcal{E}} = \frac{8}{3} \left(\frac{V}{c}\right)^2, \quad \frac{\overline{(\Delta\mathcal{E})^2}}{\mathcal{E}^2} = \frac{4}{3} \left(\frac{V}{c}\right)^2, \quad (2.4.8)$$

where the last equality follows from $\Delta\mathcal{E}^2 \simeq 4\mu^2(V/c)^2 \mathcal{E}^2$ and $\overline{\mu^2} = 1/3$. This means that the particles will not only increase their mean energy, but will also rapidly *disperse* in energy. This dispersion can be thought of as a random walk in along the energy axis, where a particle randomly takes forward- and backward steps in energy, with the step size equal to

$$|\Delta\mathcal{E}| = \Delta\mathcal{E}_{\text{rms}} \simeq \left(\overline{(\Delta\mathcal{E})^2} \right)^{1/2}, \quad (2.4.9)$$

and with the probability of a forward or backward step almost equal. Such a random walk can be characterised by an energy diffusion coefficient $\mathcal{D}_{\mathcal{E}}$ that is defined as

$$\begin{aligned} \mathcal{D}_{\mathcal{E}} &\equiv \frac{\overline{(\Delta\mathcal{E})^2}}{2\Delta t} = \frac{2c}{3\lambda} \left(\frac{V}{c}\right)^2 \mathcal{E}^2 \\ &= \frac{1}{4}\alpha \mathcal{E}^2. \end{aligned} \quad (2.4.10)$$

Here $\Delta t = 1/\overline{\mathcal{R}_*} = c/\lambda$ is the mean time between two subsequent collisions. Since forward steps are slightly more frequent than backwards steps, the random walk has a slight bias⁵, leading to a drift towards higher energy.

One can show that this leads to a flow of particles along the energy axis, with a *flux* (the number of particles passing a given energy per second) equal to

$$S(\mathcal{E}, t) = \frac{d\mathcal{E}}{dt} \mathcal{N}(\mathcal{E}, t) - \frac{\partial}{\partial \mathcal{E}} [\mathcal{D}_{\mathcal{E}} \mathcal{N}(\mathcal{E}, t)]. \quad (2.4.11)$$

Here $\mathcal{N}(\mathcal{E}, t)$ is the number of particles per unit energy at an energy $\mathcal{E}$ at time t .

⁵From Eqn. (2.4.3) it is easily seen that the difference between the rate of head-on and overtaking collisions is of order $(V/c)\overline{\mathcal{R}_*} \ll \overline{\mathcal{R}_*}$, with $\overline{\mathcal{R}_*}$ the mean collision rate.

2.5 Energy gain in regular Fermi acceleration

To calculate the energy gain in regular Fermi acceleration it is simplest to look at the mechanical analogue of Figure 2.1, where particles are confined to the vicinity of an ideally reflecting wall by scattering centers that approach this wall. If one goes to the frame where the scattering centers are at rest⁶, and the wall is moving with velocity $-V$, the only time a particle gains energy is when it is reflected by the wall. Note that *each* collision with the wall is a head-on collision, so the relative energy gain in the rest frame of the scattering centers equals for $V \ll v \leq c$ and for specular reflection (see Eqn. 2.3.11):

$$\frac{\Delta \mathcal{E}}{\mathcal{E}} \simeq \frac{2Vv \cos \theta}{c^2} \quad (2.5.1)$$

Here θ is the angle at which the particle hits the wall: the momentum component along the direction normal to the wall is $p_{\parallel} = p \cos \theta$. If the density of energetic particles is n , and if the scattering keeps the distribution of particle velocities isotropic in the rest frame of the scattering centers, the flux of particles hitting the wall at an incidence angle θ is (neglecting a term of order V/v)

$$\mathcal{F}(\theta) \simeq nv \cos \theta \quad (\text{for } 0 \leq \theta \leq \tfrac{1}{2}\pi) . \quad (2.5.2)$$

This is the number of particles that hits the wall per unit area per second. The mean energy gain is therefore a *flux-weighted average* that is formally defined as:

$$\frac{\overline{\Delta \mathcal{E}}}{\mathcal{E}} = \frac{\frac{1}{2} \int_0^1 d\mu \mathcal{F}(\mu) (\Delta \mathcal{E}/\mathcal{E})}{\frac{1}{2} \int_0^1 d\mu \mathcal{F}(\mu)} = \frac{\frac{1}{2} \int_0^1 d\mu nv\mu^2 (2Vv/c^2)}{\frac{1}{2} \int_0^1 d\mu nv\mu} . \quad (2.5.3)$$

Here I have defined $\mu = \cos \theta$ and use the fact that for an isotropic distribution the probability of finding a particle with $\cos \theta$ in the range $\mu, \mu + d\mu$ is $\frac{1}{2}d\mu$. The two integrals over $\mu = \cos \theta$ are elementary, and one finds:

$$\frac{\overline{\Delta \mathcal{E}}}{\mathcal{E}} = \frac{4}{3} \frac{Vv}{c^2} .$$

(2.5.4)

⁶If a particle systematically gains energy in one (inertial) reference frame, it does so in all frames.

If two subsequent reflections by the wall are -on average- a timespan T_{cy} apart, the *cycle time* of a particle, the mean energy gain per unit time is

$$\frac{\overline{d\mathcal{E}}}{dt} = \frac{\overline{\Delta\mathcal{E}}}{T_{\text{cy}}} = \frac{4}{3} \left(\frac{Vv}{c^2 T_{\text{cy}}} \right) \mathcal{E} . \quad (2.5.5)$$

The cycle time T_{cy} can be estimated rather simply in the model of Figure 2.2. If the scattering centers have a density n_* a particle will travel (on average) one mean free path $\lambda = 1/n_*\sigma_*$ between two scatterings. Here σ_* is the scattering cross section. Repeated scattering leads to a diffusive motion (random walk) of the particle, which therefore only travels a typical distance $\sqrt{N}\lambda \propto \sqrt{t}$ after $N \propto t$ scatterings⁷. This is a very inefficient mode of propagation. The wall on the other hand travels steadily, covering a distance Vt in a time t . This means that particles will be swept up by the wall shortly after they start diffusing. They can not travel for more than about one mean free path before that happens. As a result, the wall overtakes particles that have just been reflected again after a time $\Delta t = T_{\text{cy}} \sim \lambda/V$, leading to a new reflection. A more precise calculation⁸ gives:

$$T_{\text{cy}} = \frac{4}{3} \frac{\lambda}{V} . \quad (2.5.6)$$

This determines the mean energy gain per unit time as:

$$\frac{\overline{d\mathcal{E}}}{dt} = \left(\frac{V^2 v}{c^2 \lambda} \right) \mathcal{E} . \quad (2.5.7)$$

For relativistic particles with $v \simeq c$ this is almost the same expression as for stochastic Fermi acceleration, see Eqn. (2.4.5). So: for a *given* velocity V and scattering mean free path λ regular and stochastic Fermi acceleration are roughly equally effective. There are two important differences, however:

1. In regular Fermi acceleration there is **no** (actually: very little) diffusion in energy;
2. In the physical realization of regular Fermi acceleration, a process known as *Diffusive Shock Acceleration* that is discussed in more detail below, the velocities involved tend to be (fractions of) the shock velocity, which can be very high.

⁷A readable introduction into diffusion processes is Ya. B. Zeldovich, A.A. Ruzmaikin & D.D. Sokoloff 1990: *The Almighty Chance*, World Scientific Publishers, Singapore.

⁸e.g. L.O'C. Drury 1983: *Rep. Prog. Phys.* **46**, 963.

Chapter 3

Building power laws

3.1 Introduction

The energy distribution of accelerated particles in astrophysical acceleration in an energy range far above the injection energy is always the result of a competition between three processes: [1] the *energy gain* due to Fermi-like acceleration that increases the (average) energy of particles in the source, [2] the *losses* (mostly radiation losses for leptons and collisional losses for hadrons) that degrade the particle energy and [3] the *escape* of particles from the region where the acceleration takes place. Generally speaking, older particles will be more energetic having been accelerated longer. But they are also less abundant since -generally speaking- only a small fraction of particles is retained in the source for a very long time: the distribution of particle ages plays an important role in determining the energy distribution. Here I will derive the equation for the energy distribution $\mathcal{N}(\mathcal{E}, t)$ of the particles inside an accelerator, and determine the circumstances under which one may expect to find a power-law with $\mathcal{N}(\mathcal{E}, t) \propto \mathcal{E}^{-s}$ as is often observed.

In the derivation I will employ a very simple approximation, known as a *Box Model*: the accelerator is treated as a Leaky Box, where particles are injected into the Box, are accelerated (possibly in competition with energy losses) until they escape from the Box. An observer outside the Box sees the distribution of the escaped particles if he measures them directly¹, as happens for instance in the case of the Galactic Cosmic Rays observed at the top of the Earth's atmosphere. If on the other hand the observer determines the distribution indirectly, for instance by observing the synchrotron radiation coming from a population of relativistic electrons inside the accelerator, he will determine the distribution of particles inside the Box. These two distributions need not be the same if the rate of escape from the Box is a function of energy (see below).

¹If the particles loose more energy *in transit* from the source to the observer, he should correct for those additional losses.

3.2 Flow equation in energy space

Let us define

$$dN = \mathcal{N}(\mathcal{E}, t) d\mathcal{E} \quad (3.2.1)$$

to be the number of particles inside the Box at time t , with their energy in the interval $\mathcal{E}, \mathcal{E} + d\mathcal{E}$. As a result of acceleration and/or energy losses, particles change their energy, which can be thought of as a flow of particles along the energy axis. Let the associated *flux* of particles along the energy axis be $S(\mathcal{E}, t)$, defined in such a manner that

$$\Delta N = S(\mathcal{E}, t) \Delta t \quad (3.2.2)$$

particles move across a point at given energy $\mathcal{E}$ on the energy axis in a time interval Δt . If particles move to higher energy one has $S(\mathcal{E}, t) > 0$. If there is a net energy loss one has $S(\mathcal{E}, t) < 0$. In the case of regular acceleration, where the particle energy systematically increases (or decreases due to overwhelming losses), one has:

$$S(\mathcal{E}, t) = \left\langle \frac{d\mathcal{E}}{dt} \right\rangle \mathcal{N}(\mathcal{E}, t). \quad (3.2.3)$$

The quantity $\langle d\mathcal{E}/dt \rangle$ is the *mean acceleration rate*, the ‘flow velocity’ along the energy axis. Assuming for simplicity $S(\mathcal{E}, t) > 0$, this definition implies that the flow along the energy axis results in a change of the number of particles in the energy interval $d\mathcal{E}$ over a time Δt equal to

$$\Delta N_{\text{flow}} = \underbrace{S(\mathcal{E}, t) \Delta t}_{\text{flowing in at } \mathcal{E}} - \underbrace{S(\mathcal{E} + d\mathcal{E}, t) \Delta t}_{\text{flowing out at } \mathcal{E} + d\mathcal{E}}. \quad (3.2.4)$$

Let fresh particles be injected into the box at a rate such that a number of fresh particles equal to

$$\Delta N_{\text{inj}} = Q(\mathcal{E}, t) d\mathcal{E} \Delta t \quad (3.2.5)$$

enter the energy interval $\mathcal{E}, \mathcal{E} + d\mathcal{E}$ in a time Δt .

Finally, let particles escape from the Box with a typical escape time $T(\mathcal{E})$ so that

$$\Delta N_{\text{esc}} = -\mathcal{N}(\mathcal{E}, t) d\mathcal{E} \frac{\Delta t}{T(\mathcal{E})} \quad (3.2.6)$$

particles escape from the interval $\mathcal{E}, \mathcal{E} + d\mathcal{E}$ in a time $\Delta t \ll T$. We can now calculate how the number of particles in the energy interval $d\mathcal{E}$ changes as a result of flow along the energy axis, injection and escape, see the Figure below:

$$\begin{aligned} \frac{\partial \mathcal{N}(\mathcal{E}, t)}{\partial t} d\mathcal{E} \Delta t &= \Delta N_{\text{flow}} + \Delta N_{\text{inj}} + \Delta N_{\text{esc}} \\ &= \left(S(\mathcal{E}, t) - S(\mathcal{E} + d\mathcal{E}, t) + Q(\mathcal{E}, t) d\mathcal{E} - \frac{\mathcal{N}(\mathcal{E}, t) d\mathcal{E}}{T(\mathcal{E})} \right) \Delta t. \end{aligned} \quad (3.2.7)$$

For infinitesimally small $d\mathcal{E}$ we can use

$$S(\mathcal{E}, t) - S(\mathcal{E} + d\mathcal{E}, t) = -\frac{\partial S(\mathcal{E}, t)}{\partial \mathcal{E}} d\mathcal{E}. \quad (3.2.8)$$

Substituting this into the above equation and canceling the common factor $d\mathcal{E} \Delta t$ we find:

$$\frac{\partial \mathcal{N}(\mathcal{E}, t)}{\partial t} = -\frac{\partial S(\mathcal{E}, t)}{\partial \mathcal{E}} + Q(\mathcal{E}, t) - \frac{\mathcal{N}(\mathcal{E}, t)}{T(\mathcal{E})}.$$

(3.2.9)

This is the generally valid flow equation in energy space that describes how the distribution of energetic particles in the Box changes.

More complete models² add additional terms that describe the effects of particle propagation in space (either inside or outside the source) to this equation. In that way one can model processes such as the diffusion and (re)acceleration of cosmic rays inside the Galaxy³.

²See for instance: M.S. Longair 1994: *High Energy Astrophysics*, 2nd Ed. Vol. 2, Ch. 21, Cambridge University Press, Cambridge, UK.

³e.g. R. Schlickeiser 2002: *Cosmic Ray Astrophysics*, Springer Verlag, Heidelberg, Germany.

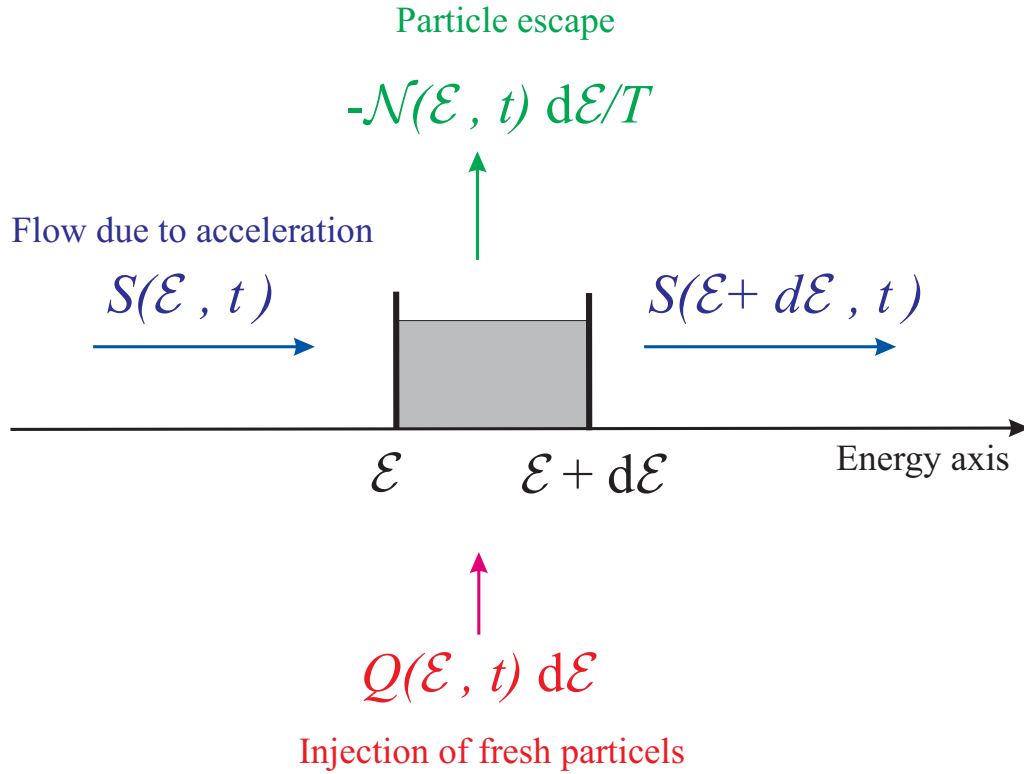


Figure 3.1: An illustration of the balance that determines how many particles can be found in an energy interval $d\mathcal{E}$. At any time, the number of particles residing in the interval equals $\mathcal{N}(\mathcal{E}, t) d\mathcal{E}$, with $\mathcal{N}(\mathcal{E}, t)$ the particle number per unit energy at energy $\mathcal{E}$. Shown are the number of particles entering per unit time ($\Delta t = 1$) due to the [1] the flux $S(\mathcal{E}, t)$ across the lower boundary at an energy $\mathcal{E}$ and [2] the injection of fresh particles as determined by the injection rate/unit energy $Q(\mathcal{E}, t)$. Particles leave the interval $d\mathcal{E}$ due to [1] the flow in energy across the upper boundary, located at an energy $\mathcal{E} + d\mathcal{E}$, and [2] due to particles escaping from the accelerator at a rate determined by the escape time T . The case shown is for acceleration, with $d\mathcal{E}/dt > 0$.

3.3 A simple example: regular Fermi acceleration with constant escape time

As an illustrative example, consider the case of regular Fermi acceleration where the energy of a particle increases exponentially in time, corresponding to an acceleration rate

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \alpha \mathcal{E} \quad (3.3.1)$$

Here $\alpha = 1/\tau_{\text{acc}}$ is the acceleration rate. As long as a particle resides in the Box the energy grows as $\mathcal{E}(t) \propto e^{\alpha t}$. Let us assume that particles are all injected with the same fixed energy $\mathcal{E}_0$ and at a fixed rate of $\mathcal{R}$ particles per second, corresponding to an injection rate $Q(\mathcal{E}, t)$ equal to:

$$Q(\mathcal{E}, t) = \mathcal{R} \delta(\mathcal{E} - \mathcal{E}_0), \quad (3.3.2)$$

where $\delta(x)$ is the Dirac delta-function. Finally we assume that the escape time T is independent of energy. In that case Eqn. (3.2.9) takes the simple form:

$$\frac{\partial \mathcal{N}(\mathcal{E}, t)}{\partial t} = -\frac{\partial}{\partial \mathcal{E}} [\alpha \mathcal{E} \mathcal{N}(\mathcal{E}, t)] + \mathcal{R} \delta(\mathcal{E} - \mathcal{E}_0) - \frac{\mathcal{N}(\mathcal{E}, t)}{T}. \quad (3.3.3)$$

Let us try to find a stationary solution where $\partial \mathcal{N}(\mathcal{E}, t)/\partial t = 0$. First look at an energy above the injection energy: $\mathcal{E} > \mathcal{E}_0$ (there are no particles with $\mathcal{E} < \mathcal{E}_0$ if $\alpha > 0$ and all particles are accelerated). The solution should satisfy:

$$\frac{\partial [\alpha \mathcal{E} \mathcal{N}(\mathcal{E})]}{\partial \mathcal{E}} + \frac{\mathcal{N}(\mathcal{E})}{T} = 0. \quad (3.3.4)$$

If substitute a power-law in energy as a trial solution,

$$\mathcal{N}(\mathcal{E}) = \kappa \mathcal{E}^{-s}, \quad (3.3.5)$$

it is easily seen that this leads to the equation

$$\kappa \left(\frac{1}{T} - \alpha (s - 1) \right) \mathcal{E}^{-s} = 0. \quad (3.3.6)$$

The power-law assumption is a good solution provided the slope s satisfies:

$$s = 1 + \frac{1}{\alpha T} . \quad (3.3.7)$$

The constant κ can be determined by first integrating the equation (3.3.3) from $\mathcal{E} = \mathcal{E}_0 - \epsilon$ to $\mathcal{E} = \mathcal{E}_0 + \epsilon$, and then taking the limit $\epsilon \downarrow 0$. Since $\mathcal{N}(\mathcal{E}) = 0$ for $\mathcal{E} < \mathcal{E}_0$ one finds, using the properties of the δ -function,

$$-\alpha \mathcal{E}_0 \mathcal{N}(\mathcal{E}_0) + \mathcal{R} = 0 . \quad (3.3.8)$$

With assumption (3.3.5) this determines κ as:

$$\kappa = \frac{\mathcal{R} \mathcal{E}_0^{s-1}}{\alpha} . \quad (3.3.9)$$

The whole solution is therefore given by:

$$\mathcal{N}(\mathcal{E}) = \frac{\mathcal{R}}{\alpha \mathcal{E}_0} \left(\frac{\mathcal{E}}{\mathcal{E}_0} \right)^{-(1+1/\alpha T)} \quad (\text{for } \mathcal{E} \geq \mathcal{E}_0) . \quad (3.3.10)$$

One sees that the slope of the distribution is determined by the quantity $1/\alpha T = \tau_{\text{acc}}/T$, the ratio of the typical (exponential) time scale for acceleration and the time scale for escape.

This result has a simple physical interpretation. Consider a group of ΔN_0 particles in an energy interval $\mathcal{E}_0$, $\mathcal{E}_0 + d\mathcal{E}_0$ at time $t = 0$:

$$\Delta N_0 = \mathcal{N}(\mathcal{E}_0) d\mathcal{E}_0 . \quad (3.3.11)$$

After a time t a fraction $e^{-t/T}$ of the particles is still inside the Box, i.e.

$$\Delta N = \mathcal{N}(\mathcal{E}) d\mathcal{E} = e^{-t/T} \mathcal{N}(\mathcal{E}_0) d\mathcal{E}_0 , \quad (3.3.12)$$

and their energy now equals $\mathcal{E} = \mathcal{E}_0 e^{\alpha t}$. This means that the energy interval in which these particles can be found has also stretched: $d\mathcal{E} = d\mathcal{E}_0 e^{\alpha t}$. Solving for $\mathcal{N}(\mathcal{E})$:

$$\mathcal{N}(\mathcal{E}) = e^{-t/T} \mathcal{N}(\mathcal{E}_0) \frac{d\mathcal{E}_0}{d\mathcal{E}} = e^{-t/T - \alpha t} \mathcal{N}(\mathcal{E}_0) = \mathcal{N}(\mathcal{E}_0) \left(\frac{\mathcal{E}}{\mathcal{E}_0} \right)^{-(1+1/\alpha T)} . \quad (3.3.13)$$

In the last step I have eliminated the elapsed time t from the equations by using the relation

$$t = \frac{1}{\alpha} \ln(\mathcal{E}/\mathcal{E}_0) . \quad (3.3.14)$$

Relation (3.3.13) is exactly the same power-law as derived above using the flux equation in energy space.

This alternate derivation shows clearly how the slope is determined: by the competition between the energy gain and particle escape from the Box. There are fewer particles at high energy since they take a longer time to reach that energy, and a larger particle fraction has escaped in that time. This explains the factor $(\mathcal{E}/\mathcal{E}_0)^{-1/\alpha T}$. The stretching of the energy interval $d\mathcal{E}$ leads to an extra factor $(\mathcal{E}/\mathcal{E}_0)^{-1}$: $d\mathcal{E}/d\mathcal{E}_0 = e^{\alpha t} = \mathcal{E}/\mathcal{E}_0$.

3.3.1 General solution for arbitrary acceleration rate and escape

The reason why such a simple power law is found in the above simple case has to do with the fact that both α and T were assumed to be energy-independent: they do not introduce an energy scale into the problem, and no specific energy is preferred for $\mathcal{E} > \mathcal{E}_0$. If on the other hand we allow the acceleration rate and the escape time to depend on energy so that

$$\alpha \implies \alpha(\mathcal{E}) \text{ and } T \implies T(\mathcal{E}) , \quad (3.3.15)$$

one can show that the flow equation (3.3.3) can be manipulated into an equation for $\tilde{n}(\mathcal{E}) \equiv \mathcal{E} \mathcal{N}(\mathcal{E})$ of the form

$$\frac{\mathcal{E}}{\tilde{n}} \frac{d\tilde{n}}{d\mathcal{E}} = \frac{d \ln \tilde{n}}{d \ln \mathcal{E}} = -q(\mathcal{E}) \quad (3.3.16)$$

for $\mathcal{E} > \mathcal{E}_0$. Here $q(\mathcal{E})$ is defined as

$$q(\mathcal{E}) = \frac{1}{\alpha(\mathcal{E})T(\mathcal{E})} + \frac{\mathcal{E}}{\alpha(\mathcal{E})} \frac{d\alpha}{d\mathcal{E}} . \quad (3.3.17)$$

The solution for the energy distribution inside the box takes the following form:

$$\mathcal{N}(\mathcal{E}) = \frac{\tilde{n}}{\mathcal{E}} = \frac{K}{\mathcal{E}} e^{-Q(\mathcal{E}, \mathcal{E}_0)} . \quad (3.3.18)$$

Here K a constant (determined below) and the function $Q(\mathcal{E}, \mathcal{E}_0)$ is defined as:

$$\begin{aligned} Q(\mathcal{E}, \mathcal{E}_0) &= \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{\mathcal{E}'} q(\mathcal{E}') = \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{\alpha(\mathcal{E}')\mathcal{E}'} \left(\frac{1}{T(\mathcal{E}')} + \mathcal{E}' \frac{d\alpha}{d\mathcal{E}'} \right) \\ &= \ln \left(\frac{\alpha(\mathcal{E})}{\alpha(\mathcal{E}_0)} \right) + \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{\dot{\mathcal{E}}' T(\mathcal{E}')} . \end{aligned} \quad (3.3.19)$$

Here I have written $\dot{\mathcal{E}}$ for the mean energy gain per unit time:

$$\dot{\mathcal{E}}(\mathcal{E}) \equiv \alpha(\mathcal{E}) \mathcal{E} = \left\langle \frac{d\mathcal{E}}{dt} \right\rangle . \quad (3.3.20)$$

This means that solution (3.3.18) can be re-expressed as

$$\mathcal{N}(\mathcal{E}) = \frac{\alpha_0 K}{\dot{\mathcal{E}}(\mathcal{E})} \exp \left(- \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{\dot{\mathcal{E}}' T(\mathcal{E}')} \right) \quad (3.3.21)$$

with $\alpha_0 \equiv \alpha(\mathcal{E}_0)$ and $\dot{\mathcal{E}}' \equiv \dot{\mathcal{E}}(\mathcal{E}')$. The exponential term gives the effect of particle escape, where particles of energy $\mathcal{E}$ take a time $dt = d\mathcal{E}/\dot{\mathcal{E}}$ to gain an small amount of energy $d\mathcal{E}$, while suffering a reduction in particle numbers due to escape by a factor $e^{-dt/T(\mathcal{E})} = e^{-d\mathcal{E}/\dot{\mathcal{E}} T(\mathcal{E})}$. Multiplying all the escape factors over a finite time interval (in particular: the time needed to reach an energy $\mathcal{E}$ for a starting energy $\mathcal{E}_0$) leads to the exponent with the integral.

The constant K is easily determined by integration of the flow equation across $\mathcal{E} = \mathcal{E}_0$. One finds $K = \mathcal{R}/\alpha_0$ and the final form of the solution is

$$\mathcal{N}(\mathcal{E}) = \frac{\mathcal{R}}{\dot{\mathcal{E}}(\mathcal{E})} \exp \left(- \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{\dot{\mathcal{E}}' T(\mathcal{E}')} \right) .$$

(3.3.22)

This general solution will only lead to a power law in energy if $Q = a + b \ln \mathcal{E}$ with a and b constants, which in turn implies that $q(\mathcal{E})$ must be a constant.

That means that the energy dependence of α and T must be ‘fine tuned’ to each other, for example:

$$\alpha(\mathcal{E}) = \alpha_0 \left(\frac{\mathcal{E}}{\mathcal{E}_0} \right)^\mu, \quad T(\mathcal{E}) = T_0 \left(\frac{\mathcal{E}}{\mathcal{E}_0} \right)^{-\mu}, \quad q = \frac{1}{\alpha_0 T_0} + \mu. \quad (3.3.23)$$

In this example the ratio of the acceleration time $\tau_{\text{acc}} = 1/\alpha$ and the escape time T is a constant, and the distribution inside the Box is given by

$$\mathcal{N}(\mathcal{E}) = \frac{\mathcal{R}}{\alpha_0 \mathcal{E}_0} \left(\frac{\mathcal{E}}{\mathcal{E}_0} \right)^{-s} \text{ with } s = 1 + \frac{1}{\alpha_0 T_0} + \mu. \quad (3.3.24)$$

Generally speaking such a fine-tuning between the escape- and the acceleration time will not occur, and the resulting spectrum (3.3.22) is not a power-law in energy: when one plots $\ln \mathcal{N}(\mathcal{E}, t)$ vs. $\ln \mathcal{E}$ the resulting graph will be curved. It is easy to show that the above example is in fact the *only* class of simple scalings of $\alpha = 1/\tau_{\text{acc}}$ and T with energy where a power-law results. The simple case of constant α and T treated above corresponds to the choice $\mu = 0$.

If the escape time depends on energy, the distinction between the energy distribution inside the accelerator and that of the escaping particles becomes important: an outside observer that *directly* measures particles from the accelerator measures the flux of *escaping* particles. This particle flux $\mathcal{F}(\mathcal{E})$ scales as

$$\mathcal{F}(\mathcal{E}) \propto \left(\frac{d\mathcal{N}(\mathcal{E})}{dt} \right)_{\text{esc}} = \frac{\mathcal{N}(\mathcal{E})}{T(\mathcal{E})}. \quad (3.3.25)$$

This means that the outside particle flux scales as:

$$\mathcal{F}(\mathcal{E}) \propto \frac{\mathcal{R}}{T(\mathcal{E})\dot{\mathcal{E}}(\mathcal{E})} \exp \left(- \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{\dot{\mathcal{E}}' T(\mathcal{E}')} \right). \quad (3.3.26)$$

To be specific: a detector with a collecting area $\mathcal{A}$ at a distance D from an isotropically radiating source will collect a number of particles per unit time at energy $\mathcal{E}$ equal to

$$\left(\frac{dN}{dt d\mathcal{E}} \right)_{\text{obs}} = \frac{\mathcal{A}}{4\pi D^2} \left(\frac{\mathcal{R}}{T(\mathcal{E})\dot{\mathcal{E}}(\mathcal{E})} \right) \exp \left(- \int_{\mathcal{E}_0}^{\mathcal{E}} \frac{d\mathcal{E}'}{\dot{\mathcal{E}}' T(\mathcal{E}')} \right) \quad (3.3.27)$$

if the particles do not loose a substantial amount of energy in transit.

For the simple scaling (3.3.23), where $T \propto \mathcal{E}^{-\mu}$, one finds:

$$\left(\frac{dN}{dt d\mathcal{E}} \right)_{\text{obs}} \propto \left(\frac{\mathcal{E}}{\mathcal{E}_0} \right)^{-s'} \text{ with } s' = s - \mu = 1 + \frac{1}{\alpha_0 T_0} . \quad (3.3.28)$$

3.4 Acceleration by waves

Fermi's original suggestion, where the acceleration is attributed to molecular clouds, has been superseded by a closely related mechanism: acceleration by resonant plasma waves. The term 'resonant' in this context means that the mechanism picks out those waves that have a *constant phase* as seen by an observer moving with the particle. For plane waves the phase S enters the amplitude, which varies as $\cos S$ or $\sin S$, so a constant phase 'freezes' the sinusoidal variation. In the simplest case, where there is no ambient magnetic field, waves with wavevector $\mathbf{k} = k \hat{\mathbf{k}}$, (where $k = 2\pi/\lambda$ is the wavenumber of the waves, λ their wavelength and $\hat{\mathbf{k}}$ is the unit vector in the direction of propagation) will interact with a charge provided that

$$v_{\parallel} = v_{\text{ph}} = \frac{\omega}{k} . \quad (3.4.1)$$

Here $v_{\parallel} \equiv \mathbf{v} \cdot \hat{\mathbf{k}}$ is the component of the particle velocity along the direction of propagation of the wave, and v_{ph} is the phase velocity of the wave. If this condition is satisfied, the phase of the wave does not vary in the reference frame moving with the particle, see below. Usually this resonance condition is written in a different (but equivalent) way:

$$\omega - \mathbf{k} \cdot \mathbf{v} = 0 . \quad (3.4.2)$$

If a magnetic field is present, the gyration of a particle around the field with gyration frequency $\Omega_g = qB/\gamma mc$ opens the possibility of additional *gyroresonances* at the harmonics of the gyration frequency. In that case condition (3.4.3) is modified to:

$$\omega - k_{\parallel} v_{\parallel} = n \Omega_g , \quad (3.4.3)$$

where $n = \dots - 1, 0, +1, \dots$ is an integer. In this case the subscript $\parallel$ refers to the components of $\mathbf{v}$ and $\mathbf{k}$ along the magnetic field, which is the direction in which a particle simply 'slides' along the field line with constant velocity $v_{\parallel}$.

3.4.1 The simplest possible case: one-dimensional acceleration by wave electric fields

To illustrate how this resonance comes about, I will consider the simplest possible example where a charge q of mass m moves in one dimension (denoted by x), and feels the fluctuating electric field of a plasma wave. Any plasma supports a wide array of waves, both *electrostatic waves*, where there is no magnetic field associated with the wave, and *electromagnetic waves*, where the wave has both an electric and a magnetic field⁴.

The equation of motion of the charge is

$$\frac{dp}{dt} = q E(x, t), \quad (3.4.4)$$

with $p = \gamma mv$ the particle momentum and $v = dx/dt$. A formal integration gives the momentum change over a time Δt as:

$$\Delta p = q \int_0^{\Delta t} dt E(x(t), t). \quad (3.4.5)$$

Note that we have to integrate along the orbit $x(t)$ of the particle, which is essentially unknown since it is part of the problem we are trying to solve! It is obvious that we will have to make some approximation in order to proceed further. One often uses the so-called *quasi-linear approximation*, where one employs the *unperturbed orbit* of the particle rather than the exact orbit when evaluating the time-integral in (3.4.5). This assumes implicitly that the change in velocity and momentum due to the interaction between charge and wave remains small, $\Delta p \ll p$. Without the wave electric field, and in the absence of other forces, the motion of the particle is simply:

$$x = x_0 + vt \quad (3.4.6)$$

with v the *constant* unperturbed velocity⁵. In the quasi-linear approximation one replaces the exact (but unmanageable) result by:

$$\Delta p \simeq q \int_0^{\Delta t} dt E(x_0 + vt, t). \quad (3.4.7)$$

⁴A good introduction to the subject is: R.M. Kulsrud 2005: *Plasma Physics for Astrophysics*, Chapters 9 through 11, Princeton University Press, Princeton, USA.

⁵One of the reasons why gyroresonances occur for charges in a magnetic field is the fact that the unperturbed velocity v is **not** constant due to the gyration around the magnetic field.

The electric field of a wave can be represented by a Fourier integral of the form:

$$E(x, t) = \int \frac{dk d\omega}{(2\pi)^2} \tilde{E}(k, \omega) e^{ikx - i\omega t} . \quad (3.4.8)$$

This is a mathematically consistent way of superimposing plane waves with a wide range of frequencies and wavelengths. Substituting this Fourier integral into (3.4.7) one finds:

$$\Delta p \simeq q \int \frac{dk d\omega}{(2\pi)^2} \tilde{E}(k, \omega) \int_0^{\Delta t} dt e^{iS(t)} , \quad (3.4.9)$$

where

$$S(t) \equiv kx_0 + (kv - \omega)t \quad (3.4.10)$$

is the phase of the wave as seen by the particle in the unperturbed orbit. At this level of description it becomes already obvious why the resonance condition occurs: since $e^{iS} = \cos S + i \sin S$ the time-integral vanishes unless the phase S is stationary so that

$$\frac{dS}{dt} = kv - \omega = 0 . \quad (3.4.11)$$

That is **not** the whole story, however. If there is a collection of waves with electric fluctuations, the electric field averaged over the whole ensemble of waves must vanish:

$$\langle E(x, t) \rangle = 0 . \quad (3.4.12)$$

Here I use the $\langle E \rangle$ notation to denote an ensemble average. This result is simply the consequence of the fact that an field amplitude $-E$ is equally likely to occur in an ensemble with zero mean as an amplitude E . This immediately implies that the average momentum gain vanishes:

$$\langle \Delta p \rangle = q \int_0^{\Delta t} dt \langle E(x(t), t) \rangle = 0 . \quad (3.4.13)$$

Here I assume that the time-integration and taking the ensemble average are commuting operations, which is correct if the statistical properties of the ensemble of waves do not change over time and position: a *uniform and stationary* ensemble. This result simply states that the momentum change Δp can be positive or negative with equal probability.

The following quantity does **not** vanish:

$$\langle (\Delta p)^2 \rangle = q^2 \int_0^t dt \int_0^t dt' \langle E(x, t) E(x', t') \rangle , \quad (3.4.14)$$

simply because $(\Delta p)^2$ is always positive (or zero). Here I have applied relation (3.4.5) twice, and taken the ensemble average. I simply write x for $x(t)$ and x' for $x(t')$. For each of the two electric fields in this double integral one must substitute a Fourier integral. This leads to an enormously complicated expression with two Fourier integrals, one over (k, ω) for $E(x, t)$ and one over (k', ω') for $E(x', t')$. Luckily, one is helped by the fact that for a stationary and uniform (homogeneous) ensemble the following relation holds⁶ for the Fourier components of the electric field:

$$\langle \tilde{E}(k, \omega) \tilde{E}(k', \omega') \rangle = (2\pi)^2 |\mathbf{E}|^2(k, \omega) \delta(k + k') \delta(\omega + \omega') . \quad (3.4.15)$$

The quantity $|\mathbf{E}|^2(k, \omega)$ is a measure of the electrical energy density carried by the waves: one has

$$\text{energy density} \equiv \frac{\langle E^2 \rangle}{8\pi} = \int \frac{dk d\omega}{(2\pi)^2} \frac{|\mathbf{E}|^2(k, \omega)}{8\pi} . \quad (3.4.16)$$

The two δ -functions in (3.4.15) allow one to perform one of the two Fourier integrals in a trivial fashion, say the one over (k', ω') . One is left with a single Fourier integral over (k, ω) :

$$\langle (\Delta p)^2 \rangle = q^2 \int_0^{\Delta t} dt \int_0^{\Delta t} dt' \int \frac{dk d\omega}{(2\pi)^2} |\mathbf{E}|^2(k, \omega) e^{i\overline{S}(t, t')} . \quad (3.4.17)$$

Here

$$\overline{S}(t, t') = k(x(t) - x(t')) - \omega(t - t') . \quad (3.4.18)$$

For a stationary and homogeneous ensemble $|\mathbf{E}|^2(k, \omega)$ does not depend on position x or time t , and we can reverse the order of integration, putting the Fourier integral in front.

⁶This follows from the fact that in a stationary, homogeneous ensemble the two-point correlation function $\langle E(x, t) E(x', t') \rangle$ can only depend on space and time through $\Delta x = x - x'$ and $\Delta t = t - t'$.

This leads to:

$$\langle (\Delta p)^2 \rangle = q^2 \int \frac{dk d\omega}{(2\pi)^2} |E|^2(k, \omega) \int_0^{\Delta t} dt \int_0^{\Delta t} dt' \exp(i k [x(t) - x(t')] - i \omega [t - t']) . \quad (3.4.19)$$

In the quasi-linear approximation one has

$$k [x(t) - x(t')] - \omega [t - t'] = (kv - \omega) (t - t') , \quad (3.4.20)$$

and the integrand in the double time integral is a function of the time difference $\tau \equiv t - t'$. In that case one can use the following relation that is valid for any function $F(\tau) = F(t - t')$:

$$\int_0^{\Delta t} dt \int_0^{\Delta t} dt' F(\tau) = 2 \int_0^{\Delta t} dt \int_0^t d\tau F(\tau) \simeq 2\Delta t \int_0^\infty d\tau F(\tau) . \quad (3.4.21)$$

The first equality in this chain is exact, the second one is an approximation that assumes that the τ -integration can be extended to infinity without introducing a large error. One can show that this is a reasonable approximation in the problem at hand. This implies that one can define a *momentum diffusion coefficient* by:

$$\mathcal{D}_p \equiv \frac{\langle (\Delta p)^2 \rangle}{2\Delta t} = q^2 \int \frac{dk d\omega}{(2\pi)^2} |E|^2(k, \omega) g(k, \omega) , \quad (3.4.22)$$

where the *propagator* $g(k, \omega)$ is formally defined as:

$$g(k, \omega) \equiv \int_0^\infty d\tau e^{i(kv - \omega)\tau} . \quad (3.4.23)$$

To evaluate (and regularize) the integral over τ one must make the replacement

$$\omega \implies \omega - i\sigma \quad (3.4.24)$$

with $\sigma \ll |\omega|$ a positive quantity.

We will get rid of σ again later by taking the limit $\sigma \downarrow 0$. This replacement leads to

$$e^{-i\omega\tau} \implies e^{-i\omega\tau} e^{-\sigma\tau}, \quad (3.4.25)$$

and the exponent in the integral dies away for large τ . The integral is now elementary, and one finds:

$$g(k, \omega) = \frac{-i}{\omega - kv - i\sigma}. \quad (3.4.26)$$

This expression is equivalent with⁷:

$$g(k, \omega) = \frac{\sigma}{(\omega - kv)^2 + \sigma^2} - i \frac{\omega - kv}{(\omega - kv)^2 + \sigma^2} \quad (3.4.27)$$

When this is substituted into the Fourier integral in (3.4.22) the imaginary part of this expression for $g(k, \omega)$ does not contribute to the integral: upon integration it vanishes identically. The reason is that⁸ $|\mathbf{E}|^2(k, \omega) = |\mathbf{E}|^2(-k, -\omega)$, $\text{Im}(g(k, \omega)) = -\text{Im}(g(-k, -\omega))$, while the integration extends over $-\infty < \omega < +\infty$ and $-\infty < k < +\infty$. So the integrand for the imaginary part is uneven under replacement $\omega \rightarrow -\omega$, $k \rightarrow -k$. In contrast: the real part of $g(k, \omega)$ is an even function under this replacement.

If one now uses the mathematical identity (one of the many possible definitions for the δ -function floating around in the mathematical literature)

$$\lim_{\sigma \downarrow 0} \left(\frac{\sigma}{(\omega - kv)^2 + \sigma^2} \right) = \pi \delta(\omega - kv), \quad (3.4.28)$$

the final result for the momentum diffusion coefficient is:

$$\mathcal{D}_p = q^2 \int \frac{dk d\omega}{(2\pi)^2} |\mathbf{E}|^2(k, \omega) \pi \delta(\omega - kv). \quad (3.4.29)$$

The δ -function selects precisely those waves that are resonant in the sense explained above (see Eqn. 3.4.3), the result that we anticipated on physical grounds.

⁷Just multiply both numerator and denominator in the expression for $g(k, \omega)$ by $\omega - kv + i\sigma$.

⁸This relation comes from the fact that $E(x, t)$ is a real quantity, which implies $\tilde{E}^*(k, \omega) = \tilde{E}(-k, -\omega)$ with $\tilde{E}^*$ the complex conjugate of $\tilde{E}$, and from $|\tilde{E}|^2 = \tilde{E}\tilde{E}^*$.

3.4.2 Momentum diffusion in one and three dimensions

In the one-dimensional calculation of the previous Section the existence of a well-defined momentum diffusion coefficient $\mathcal{D}_p$ implies that the momentum distribution of particles, defined in such a manner that $\mathcal{N}(p, t) dp$ is the number of particles in the momentum interval $p, p + dp$ at time t , satisfies a diffusion equation of the form⁹:

$$\frac{\partial \mathcal{N}}{\partial t} = \frac{\partial}{\partial p} \left(\mathcal{D}_p \frac{\partial \mathcal{N}}{\partial p} \right) . \quad (3.4.30)$$

In real life, with three spatial dimensions, one has to take account of the fact that the number of particles in a magnitude of momentum interval dp equals

$$dN = 4\pi p^2 F(p, t) dp , \quad (3.4.31)$$

since the relevant momentum volume is the volume of a layer of thickness dp on a momentum sphere with radius p and surface area $4\pi p^2$. Here $F(p, t)$ is the number of particles per unit momentum volume d^3p at time t . In that case, the above momentum diffusion equation is replaced by a slightly more complicated expression:

$$\frac{\partial F}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left(p^2 \mathcal{D}_p \frac{\partial F}{\partial p} \right) . \quad (3.4.32)$$

This replacement, and in particular the need for the two factors p^2 , can be understood by noting that momentum diffusion by itself does not change the number of particles: it simply redistributes the positions of the individual particles in momentum space. The process should therefore satisfy particle conservation. Using the diffusion equation one has:

$$\frac{dN}{dt} = 0 = \int_0^\infty dp 4\pi p^2 \frac{\partial F}{\partial t} = 4\pi p^2 \mathcal{D}_p \frac{\partial F}{\partial p} \Big|_0^\infty , \quad (3.4.33)$$

which is true provided $p^2 \mathcal{D}_p (\partial F / \partial p) = 0$ for $p \rightarrow \infty$. This is always the case for physically relevant conditions. In the three-dimensional case we also have to slightly modify our expression for the momentum diffusion coefficient. It becomes:

$$\mathcal{D}_p = q^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} |\mathbf{E}_\parallel|^2 \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}) . \quad (3.4.34)$$

⁹The fact that $\mathcal{D}_p$ is placed after the first momentum derivative is dictated by particle conservation.

Note that the integration over wave vectors is now also fully three-dimensional, assuming that the wave vector $\mathbf{k}$ can have any orientation. We have also replaced kv by $\mathbf{k} \cdot \mathbf{v}$, cf. Eqn. (3.4.3). Finally, $|\mathbf{E}_{\parallel}|^2$ refers to the component of the wave electric field along the direction of the particle momentum.

Generally speaking, the electric field vector $\mathbf{E}$ will have three components. This means that the Fourier integral (3.4.8) becomes a relation between vectors,

$$\mathbf{E}(\mathbf{x}, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{\mathbf{E}}(\mathbf{k}, \omega) \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t), \quad (3.4.35)$$

and relation (3.4.15) is replaced by a tensor equation of the type:

$$\langle \tilde{E}_i(k, \omega) \tilde{E}_j(k', \omega') \rangle = (2\pi)^2 (\mathbf{E} \mathbf{E})_{ij}(k, \omega) \delta(k + k') \delta(\omega + \omega'). \quad (3.4.36)$$

Here $i, j = 1, 2, 3$ are the indices enumerating the three components of the vector $\tilde{\mathbf{E}}$, and $\mathbf{E} \mathbf{E}$ is a 3×3 *dyadic tensor*¹⁰ (the correlation tensor for the electric field) which contains as its components all 9 possible ensemble-averaged combinations (i, j) of the three components of the electric field Fourier vector $\tilde{\mathbf{E}}$. If the momentum direction corresponds to a unit vector $\hat{\mathbf{n}}$ (so that $\mathbf{p} = p \hat{\mathbf{n}}$) we have, suppressing the dependence on ω and $\mathbf{k}$:

$$|\mathbf{E}_{\parallel}|^2 = \hat{n}_i (\mathbf{E} \mathbf{E})_{ij} \hat{n}_j = \hat{\mathbf{n}} \hat{\mathbf{n}} : \mathbf{E} \mathbf{E}. \quad (3.4.37)$$

The force due to the components of the electric field that are perpendicular to the particle momentum vector $\mathbf{p}$ will deflect the charge, but (to lowest order) does not change the magnitude of the momentum. This leads to scattering rather than acceleration. We do not treat the effect of this deflection here. Relation (3.4.37) allows us to write to momentum diffusion coefficient as:

$$\mathcal{D}_p = q^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} (\hat{\mathbf{n}} \hat{\mathbf{n}} : \mathbf{E} \mathbf{E})(k, \omega) \pi \delta(\omega - \mathbf{k} \cdot \mathbf{v}). \quad (3.4.38)$$

¹⁰For a quick introduction to tensors see: G.B. Arfken & H.J. Weber, *Mathematical Methods for Physicists*, Sixth Ed., Ch. 2, Elsevier Academic Press, Amsterdam, 2005. A *dyadic tensor* is formed from two vectors $\mathbf{A}$ (components A_i) and $\mathbf{B}$ (components B_j) as a *direct product* of the form $\mathbf{AB}$ with components $(\mathbf{AB})_{ij} = A_i B_j$.

3.4.3 Momentum diffusion and the acceleration rate

Momentum diffusion equation (3.4.32) can be written in a way analogous to Eqn. (3.2.9), the flow equation in energy space. Define

$$\mathcal{N}(p, t) = 4\pi p^2 F(p, t) \quad (3.4.39)$$

so that $dN = \mathcal{N}(p, t) dp$ is the number of particles in a momentum interval $p, p + dp$. By multiplying relation (3.4.32) by $4\pi p^2$ and performing some algebraic manipulations (mainly the shifting of momentum derivatives making use of the chain rule for differentiation) it is possible to cast the resulting equation in the form:

$$\frac{\partial \mathcal{N}(p, t)}{\partial t} = -\frac{\partial}{\partial p} \left\{ \dot{p} \mathcal{N} - \frac{\partial}{\partial p} (\mathcal{D}_p \mathcal{N}) \right\}. \quad (3.4.40)$$

Here the mean momentum gain per unit time, $\dot{p}$, is related to the momentum diffusion coefficient $\mathcal{D}_p$ by:

$$\dot{p}(p, t) \equiv \frac{1}{p^2} \frac{\partial}{\partial p} (p^2 \mathcal{D}_p) . \quad (3.4.41)$$

Relation (3.4.40) is the flow equation in momentum space, to which one can add terms describing energy losses, or the injection and escape of particles from the accelerator. The main conclusion is that momentum diffusion will lead to a net momentum gain (and not just dispersion of momentum) if $\dot{p}$ does not vanish.

Known results are easily recovered from this formulation: for relativistic particles with $\mathcal{E} \simeq cp$ and energy diffusion coefficient $\mathcal{D}_{\mathcal{E}}$ (see Eqn. 2.4.10), the momentum diffusion coefficient in classical Fermi acceleration with acceleration time $\tau_{\text{acc}} = 1/\alpha$ is simply

$$\mathcal{D}_p = \frac{1}{c^2} \mathcal{D}_{\mathcal{E}} = \frac{1}{4} \alpha p^2, \quad (3.4.42)$$

and the associated acceleration rate is

$$\dot{p} = \frac{1}{p^2} \frac{\partial}{\partial p} \left(\frac{1}{4} \alpha p^4 \right) = \alpha p. \quad (3.4.43)$$

This shows explicitly how Fermi's 'old' theory and the more modern incarnations that employ plasma waves are closely related. A full account of these modern versions of Fermi Acceleration can be found in the books by Melrose and Kulsrud¹¹.

The Figure below shows an example of stochastic Fermi acceleration for $\mathcal{D}_p = \alpha p^2/4$.

¹¹D.B. Melrose 1980: *Plasma Astrophysics: Nonthermal Processes in Diffuse Magnetized Plasmas*, Vol. I & II, Gordon and Breach Publishers, New York, USA;
R.M. Kulsrud 2005: *Plasma Physics for Astrophysics*, Princeton University Press, Princeton, USA, in particular Ch. 11 & 12.

Stochastic Fermi Acceleration

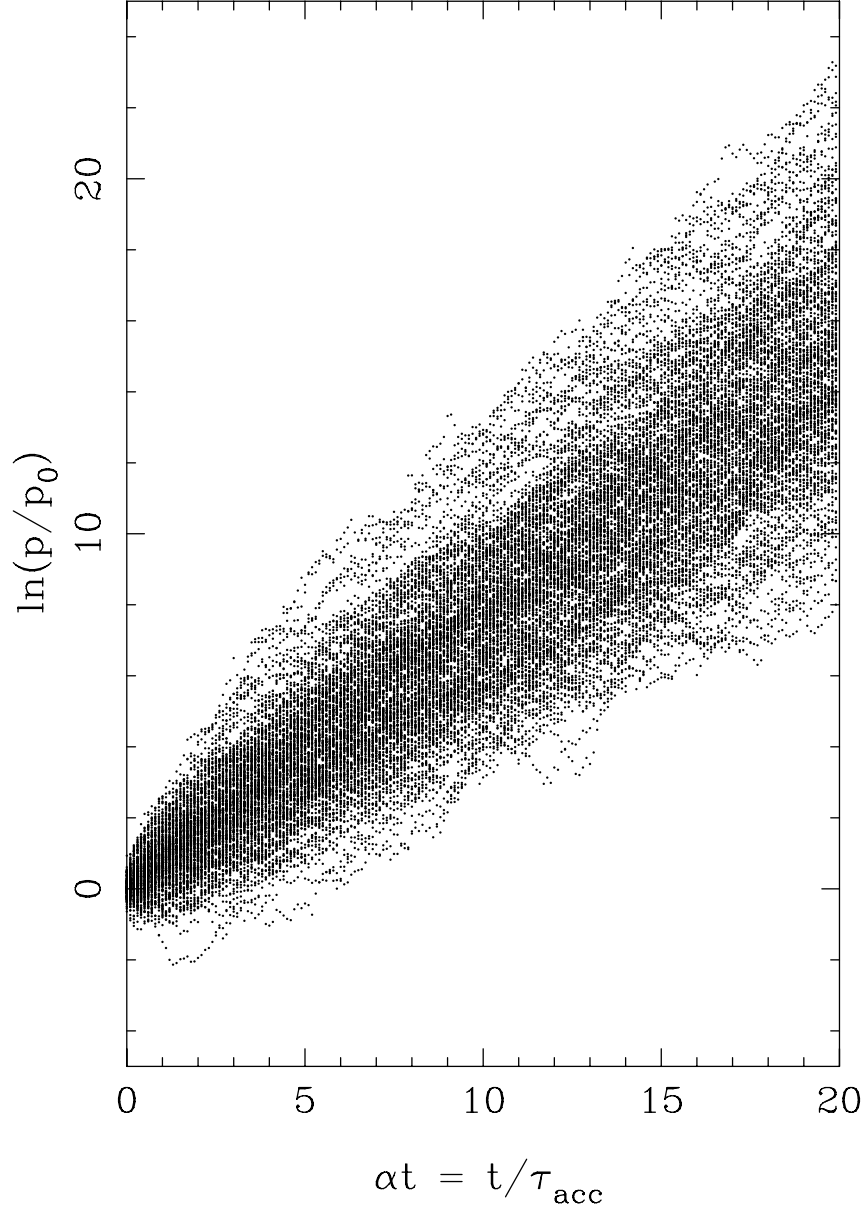


Figure 3.2: *Stochastic Fermi acceleration in the case where $\mathcal{D}_p = \alpha p^2/4 = p^2/4\tau_{\text{acc}}$. Shown is the position of 200 test particles at time intervals $\Delta t = 0.1 \tau_{\text{acc}}$. All particles start at $t = 0$ in a small momentum interval $\Delta p \sim 0.25 p$ around a momentum p_0 . On average, the particles drift to higher momentum, and -at the same time- spread out in momentum as time progresses.*

Chapter 4

Shocks as accelerators

Shocks were first considered in some detail as a possible source of particle acceleration by Colgate & Johnson¹ in the context of cosmic ray production by supernova remnants. Their idea was a simple one: a shock that is propagating down a steep density gradient, as exists in the outer layers of an exploding supernova, accelerates to near-relativistic velocities, so that the energy per particle is large. Early calculations of the interaction between charged particles and a thin shock in a magnetised plasma were done by Shatzman². These two models have not survived critical scrutiny, but paved the way for a class of models that is now considered to be the paradigm for particle acceleration in astrophysical magnetized plasmas.

4.1 Shock Drift Acceleration

Some of these first theories considered shock acceleration as a one-shot process. Consider as an example a shock propagating with velocity V_s into a magnetized plasma with a magnetic field $\mathbf{B}$. The jump in the magnetic field strength induced by the passage of the shock leads to a gain of particle energy for those particles whose gyration radius, $r_g \approx p_\perp c / qB$, is much larger than the thickness of the shock. If the shock speed V_s is much less than the particle speed v , and if the shock can be considered infinitesimally thin, the net momentum gain of a particle after interaction with the shock follows from the conservation of the magnetic moment in the encounter³,

$$\mathcal{M} = \frac{p_\perp^2}{2|\mathbf{B}|} = \text{constant} . \quad (4.1.1)$$

¹S.A. Colgate & M.H. Johnson, *Phys. Rev. Lett.* **5**, 235, 1960.

²E. Shatzman, *Ann. d'Astr.* **26**, 234.

³e.g. P.D. Hudson, *Mon. Not. R. astr. Soc.* **131**, 23, 1975.

Here $p_{\perp}$ is the component of particle momentum perpendicular to the magnetic field. The energy gain results from the field compression in the shock: unless the field is along the shock normal, the downstream field $|\mathbf{B}_{(2)}|$ exceeds the upstream field $|\mathbf{B}_{(1)}|$. One can show that if the density jumps from a value $\rho_{(1)}$ ahead of the shock to a density $\rho_{(2)} > \rho_{(1)}$ behind the shock, the components along and perpendicular to the shock normal satisfy:

$$\frac{(B_{\parallel})_{(2)}}{(B_{\parallel})_{(1)}} = 1 \quad , \quad \frac{(B_{\perp})_{(2)}}{(B_{\perp})_{(1)}} = \frac{\rho_{(2)}}{\rho_{(1)}} = r \quad . \quad (4.1.2)$$

The quantity r is known as the *shock compression factor*, and will turn out to be an important parameter in the theory of Diffusive Shock Acceleration discussed below. This means that the magnetic field strength $B = \sqrt{B_{\perp}^2 + B_{\parallel}^2}$ changes as:

$$\frac{B_{(2)}}{B_{(1)}} = \sqrt{\cos^2 \theta_B + r^2 \sin^2 \theta_B} \equiv r_B \quad , \quad (4.1.3)$$

where θ_B is the inclination angle of the magnetic field with respect to the shock normal so that $B_{\parallel(1)} = B_{(1)} \cos \theta_B$. Obviously $1 \leq r_B \leq r$ depending on the inclination angle.

This conservation law for $\mathcal{M}$ remains valid as long as the crossing event involves many gyration loops (see Figure 4.1). This process is known as *Shock Drift Acceleration* (SDA): seen from a reference frame where the shock is stationary, the particle drifts along the shock face in the direction of the flow-induced electric field, $\mathbf{E} = -(V_s \hat{\mathbf{n}} \times \mathbf{B})/c$ with $\hat{\mathbf{n}}$ the shock normal. The drift exhibited by shock-intersecting orbits results from the smaller radius of curvature (since $r_g \propto B^{-1}$) of a downstream section of a gyro-orbit compared to its upstream value.

Unless one can find a way to make the charge return to the shock, SDA is a ‘one-shot’ process: the particle crosses the shock only once. The conservation of $\mathcal{M}$ implies that the momentum gain is limited: at best (when $p_{\parallel} \simeq 0$) one has

$$p_{\text{after}} = \sqrt{r_B} p_{\text{before}} \leq 2 p_{\text{before}} \quad . \quad (4.1.4)$$

The last inequality follows from $r_B \leq r$ and $r \leq 4$ for an ideal gas (see below).

Shock Drift Acceleration does not have an equivalent in (ultra-)relativistic shocks: since $V_s \sim c$ a crossing always involves a single (small) section of a gyro-orbit, and the adiabatic limit never applies. Shock drift acceleration is essentially a single-particle collisionless mechanism, which occurs if particle orbits near the shock are undisturbed by the scattering effect of the fluctuating electromagnetic fields of hydromagnetic waves.

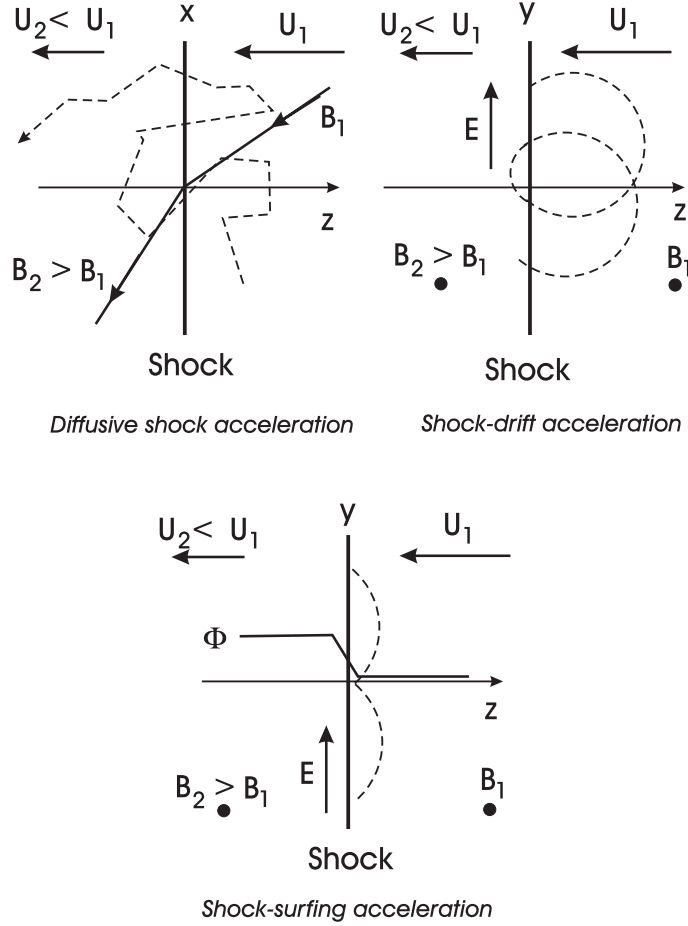


Figure 4.1: *Shock acceleration mechanisms: diffusive shock acceleration (DSA), which relies on the velocity jump created by the shock together with frequent scattering of particles near the shock, collisionless shock-drift acceleration (SDA) that relies on the jump in the magnetic field in the shock, and shock-surfing acceleration (not discussed in detail) that relies on reflection by a potential jump in the shock. Typical particle orbits are shown as dashed lines/curves, as well as the velocities U_1 (U_2) ahead (behind) the shock, the pre- and post-shock magnetic fields B_1 and B_2 , the convective electric field $E_1 = E_2 \equiv E = VB/c$ and, in the case of shock-surfing acceleration, the electrostatic potential Φ which reflects ions at the shock. The pictures are in the normal incidence frame where the shock is at rest, and the plasma flows along the shock normal, which is aligned with the z -axis. The shock has been simplified as an infinitely thin surface in the $x - y$ plane, the magnetic field is in the $x - z$ plane and the convective electric field along the y -axis. This convention is adhered to throughout these notes.*

4.2 Diffusive Shock Acceleration

In 1977/78, a number of researchers independently of each other realized that particles can be confined near a shock through the scattering action of hydromagnetic waves⁴. These particles are then accelerated due to the velocity difference between up- and downstream scattering centers. This process, known as *Diffusive Shock Acceleration* (DSA), assumes that particles have a scattering mean free path ℓ_{mfp} much larger than the shock thickness. As a result particles can cross the shock *repeatedly*.

Diffusive shock acceleration operates also for *parallel shocks*, where $\hat{n} \parallel \mathbf{B}_{(1)}$. In these shocks the magnetic field is not amplified, and the acceleration mechanism is a realization of *regular* (Fermi Type II) acceleration. A charged particle gains energy through repeated scattering between the converging up- and downstream scattering centers, much like the momentum gain experienced by table-tennis ball that is trapped in a bouncing motion between the table and a bat while the bat moves towards the table. The momentum gain, together with the escape probability at each crossing, determines the slope of the resulting spectrum (see below).

In DSA, particle gain momentum in each cycle where they cross the shock into the downstream medium, and are scattered back upstream. At the same time there is a finite escape chance per cycle: not all particles return to the shock once they enter the downstream region. The competition between the relative momentum gain $\Delta p/p$ per cycle, and the escape probability $\mathcal{P}_{\text{esc}}$ per cycle, determines the spectrum of the accelerated particles. In principle, particles can reach a high energy by completing many crossing cycles. But only a small fraction of the particles succeeds in doing so because of the finite chance of escape, once a particle finds itself downstream from the shock. Therefore, the number of particles must decay with increasing particle energy.

This situation is completely analogous with our discussion of stochastic Fermi acceleration in Section 3.3: there the spectrum also resulted from the competition between the acceleration and the escape of particles from the accelerator.

One of the main attractions of the DSA is that (in absence of losses) the spectral slope of the momentum distribution of accelerated particles does not depend on the details of the flow, such as the magnetic field orientation near the shock, or on the precise mechanism responsible for the particle diffusion that confines the particles for some time near the shock. It also naturally gives a power-law distribution in particle momentum, with a slope s which is determined solely by the *shock compression ratio* $r \equiv \rho_{(2)}/\rho_{(1)}$, with $\rho_{(1)}$ ($\rho_{(2)}$) is the density ahead (behind) the shock:

$$\mathcal{N}(p) dp \propto p^{-s} dp, \text{ with } s = \frac{r+2}{r-1}. \quad (4.2.1)$$

⁴G.E. Krinsky, *Doklady Akad. Nauk, SSR* **42**, 1306, 1977; W.I. Axford, W.I., E. Leer, & G. Skadron, *Proc. 15th Int. Cosmic Ray Conf.* (Plovdiv), **11**, 132, 1977; A.R. Bell, *Mon. Not. R. astr. Soc.* **182**, 147; R.D. Blandford & J.P. Ostriker, *Astrophys. J. Lett.* **221**, L29, 1977.

Here $dN = \mathcal{N}(p) dp$ is the number of particles found in the momentum interval p , $p + dp$. This power-law, and the relation of its slope s with the compression in the shock, will be derived below.

4.3 Physics of Diffusive Shock Acceleration

The energy gain of particles in DSA derives from the velocity difference between the up-and downstream flow. The mean momentum gain per shock crossing cycle⁵ equals (compare Eqn. 2.5.4 and use $\Delta\mathcal{E} = v \Delta p$):

$$\frac{\Delta p}{p} = \frac{4}{3} \frac{U_{(1)} - U_{(2)}}{v} . \quad (4.3.1)$$

The factor $4/3$ comes from a careful consideration of the average of $\Delta p/p$ weighed by the particle flux at different crossing angles, as discussed above. As is customary in most papers on shock acceleration, I will formulate the equations in terms of the momentum p rather than the energy $\mathcal{E}$.

Particles upstream are *always* swept up by the shock. The reason is that the scattering in the upstream medium causes the particles to *diffuse* with respect to the flow: this is an inefficient process that causes the particles to be advected into the shock by the flow. As always in this game, the scattering is provided by low-frequency magnetohydrodynamic waves, which cause irregular fluctuations in the magnetic field.

In the *downstream* flow the same holds: particles diffuse slowly with respect to the flow. But in this case the flow is directed *away* from the shock. This means that there is no guarantee that the particle will recross the shock into the upstream medium once it finds itself downstream! There is a finite escape probability downstream.

This escape probability is easily calculated for slow shock, where $U_{(2)} \ll v$ with v the particle velocity. If scattering is sufficiently strong, one may assume that the particle velocity v is distributed *isotropically* in the rest frame of the post-shock plasma. Let us denote the velocity component along the shock normal by (see figure below)

$$v_n = v \cos \theta . \quad (4.3.2)$$

⁵A shock crossing cycle takes a particle from the upstream flow into the downstream flow, and back.

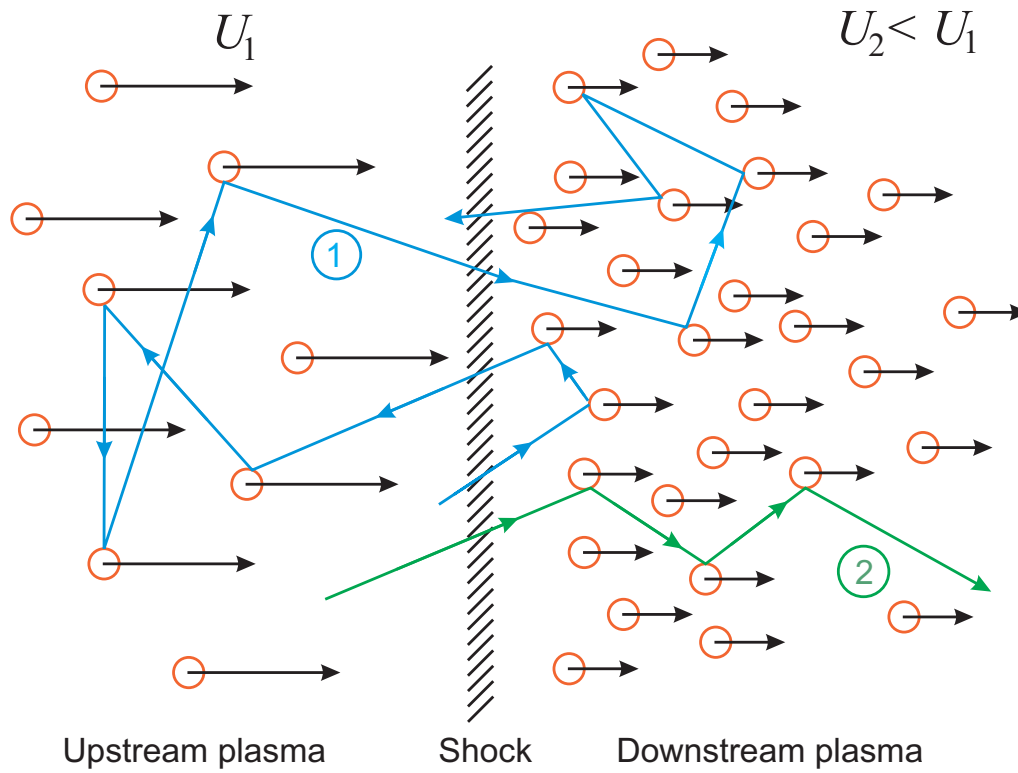


Figure 4.2: *Illustration of the physics of Diffusive Shock Acceleration. Scattering centers (in red) are advected by a flow. A shock, treated as an infinitesimally thin discontinuity, slows down the flow. In front of the shock the flow velocity is $U_{(1)}$, behind the shock it equals $U_{(2)} < U_{(1)}$. Particles are repeatedly scattered between the up- and downstream scattering centers. In doing so, they gain energy since the up- and downstream scattering centers converge. Some particles will cross the shock repeatedly (orbit 1, in blue). Ultimately, all particles will escape into the downstream flow (orbit 2, in green).*

Only those particles will re-cross the shock which have $v_n < -U_{(2)}$, where $-U_{(2)}$ is the velocity of the shock surface in the rest frame of the downstream plasma⁶. This implies that the angle θ must satisfy:

$$-1 \leq \cos \theta < -U_{(2)}/v \quad (4.3.3)$$

The total flux across the shock, back onto the upstream flow, is (for an isotropic distribution in the downstream frame) proportional to

$$\begin{aligned} \mathcal{F}_{\rightarrow u} &\propto \int_{-1}^{-U_{(2)}/v} d \cos \theta \left(v \cos \theta + U_{(2)} \right) \\ &= -\frac{\left(v - U_{(2)} \right)^2}{2v} . \end{aligned} \quad (4.3.4)$$

The minus sign reflects the fact that this flux is directed toward the left (see Figure 4.3), and the $U_{(2)}$ term enters since the shock surfaces moves with respect to the downstream plasma rest frame where the scattering keeps the particle distribution isotropic.

The flux of particles away from the shock, further into the downstream flow, is due to particles with

$$-U_{(2)}/v \leq \cos \theta \leq 1 . \quad (4.3.5)$$

This flux into the downstream flow is therefore proportional to:

$$\begin{aligned} \mathcal{F}_{\rightarrow d} &\propto \int_{-U_{(2)}/v}^1 d \cos \theta \left(v \cos \theta + U_{(2)} \right) \\ &= \frac{\left(v + U_{(2)} \right)^2}{2v} . \end{aligned} \quad (4.3.6)$$

⁶Of course this calculation assumes that the particles are not scattered on their way back to the shock, so it only applies for particles less than one mean-free-path from the shock.

The *return probability*, the chance that a particle ultimately finds its way back into the upstream flow once it is in the downstream flow, is the absolute value of the ratio of these two fluxes, which equals:

$$\mathcal{P}_{\text{ret}} = \left| \frac{\mathcal{F}_{\rightarrow \text{u}}}{\mathcal{F}_{\rightarrow \text{d}}} \right| = \left(\frac{v - U_{(2)}}{v + U_{(2)}} \right)^2. \quad (4.3.7)$$

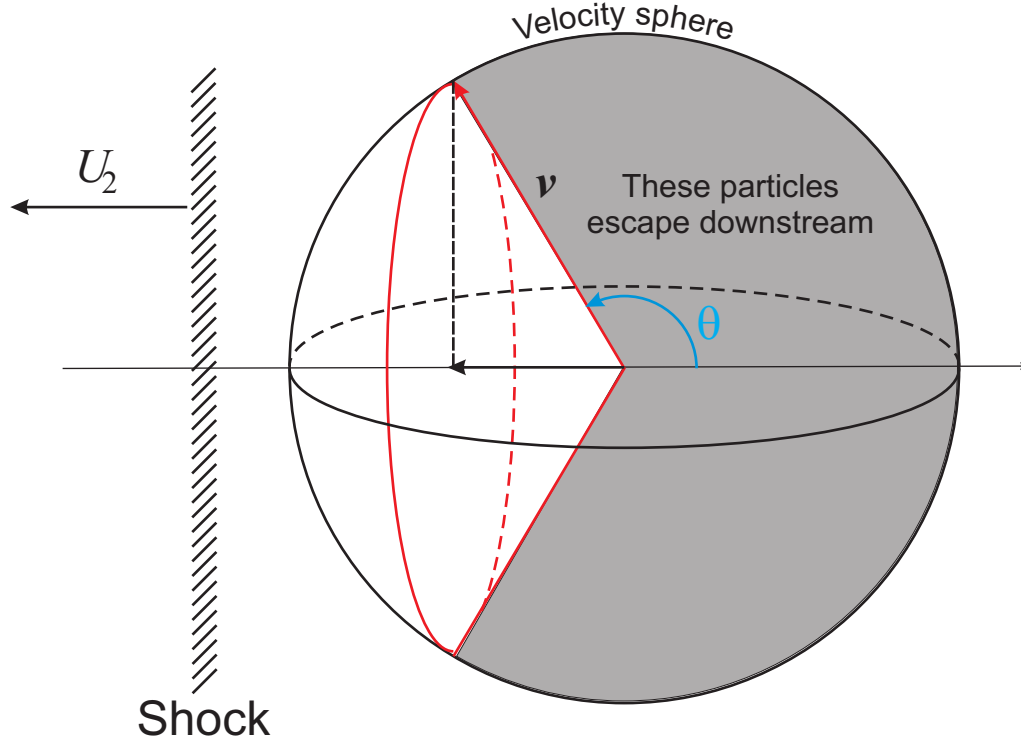


Figure 4.3: The velocities of particles in the rest frame of the downstream medium are nearly isotropically distributed over a sphere. In this frame, the shock moves with velocity U_2 towards the left. Only those particles with their velocity $\mathbf{v}$ directed within in the cone defined by $-1 \leq \cos \theta \leq -U_{(2)}/v$ (the cone bounded in red) that are within ~ 1 mean-free path from the shock can cross the shock again into the upstream flow. Particles in the gray part of the velocity sphere move away from the shock, and escape.

Assuming $v \gg U_{(2)}$ we can approximate this by:

$$\mathcal{P}_{\text{ret}} \approx 1 - \frac{4U_{(2)}}{v} \quad (4.3.8)$$

The *escape* probability per cycle,

$$\mathcal{P}_{\text{esc}} \equiv 1 - \mathcal{P}_{\text{ret}} \approx \frac{4U_{(2)}}{v}, \quad (4.3.9)$$

is small in this case. Therefore, a particle typically completes many shock crossing cycles before escaping into the far downstream flow.

4.3.1 The spectrum resulting from DSA

The spectrum that results from DSA can be calculated in a simple fashion from the mean momentum gain (4.3.1) and the escape probability (4.3.9). Let there be $\mathcal{N}(p) dp$ particles in a momentum interval dp_0 around momentum p_0 . After completing one cycle, these particles are at a momentum $p_0 + \Delta p$, and a fraction $\mathcal{P}_{\text{ret}} = 1 - 4U_{(2)}/v$ of the particles actually completes the cycle. The rest escapes. Since the number of particles that crosses the shock is proportional to the flux $d\mathcal{F} \propto v \mathcal{N}(p) dp$, and assuming that no new particles are added, one must have⁷:

$$(v \mathcal{N}(p) dp)_{p=p_0+\Delta p} = \left(1 - \frac{4U_{(2)}}{v}\right) v \mathcal{N}(p_0) dp_0. \quad (4.3.10)$$

Due to the momentum change, the momentum interval dp in which the particles can be found now equals:

$$dp = d(p_0 + \Delta p) = \left(1 + \frac{d\Delta p}{dp}\right)_{p_0} dp_0. \quad (4.3.11)$$

From this point on, I will drop the subscript 0 from p_0 , since this argument is valid at any momentum p . Substituting (4.3.11) into (4.3.10), the latter equation can be written as

$$(v\mathcal{N})_{p+\Delta p} \left(1 + \frac{d\Delta p}{dp}\right)_p dp = \left(1 - \frac{4U_{(2)}}{v}\right) v \mathcal{N}(p) dp. \quad (4.3.12)$$

Assuming that $\Delta p \ll p$ (or equivalently: $U_{(1)} - U_{(2)} \ll v$) one can expand to first order in Δp :

$$(v\mathcal{N})_{p+\Delta p} \approx v\mathcal{N}(p) + \frac{\partial(v\mathcal{N})}{\partial p} \Delta p. \quad (4.3.13)$$

⁷Remember that the velocity v is a function of the particle momentum p !

In that case, it is easily seen that the momentum distribution $\mathcal{N}(p)$ must satisfy the equation

$$\frac{\partial}{\partial p} (v \Delta p \mathcal{N}(p)) = - \left(\frac{4U_{(2)}}{v} \right) v \mathcal{N}(p) , \quad (4.3.14)$$

neglecting terms of order $(\Delta p)^2$. Using expression (4.3.1) for Δp , one immediately finds:

$$\frac{\partial}{\partial p} (p \mathcal{N}(p)) = - \left(\frac{3U_{(2)}}{U_{(1)} - U_{(2)}} \right) \mathcal{N}(p) . \quad (4.3.15)$$

The solution is a simple power-law in momentum:

$$\mathcal{N}(p) = \kappa p^{-s} \text{ with } s = \frac{U_{(1)} + 2U_{(2)}}{U_{(1)} - U_{(2)}} . \quad (4.3.16)$$

If one defines the *shock compression ratio* r ,

$$r = \frac{\rho_{(2)}}{\rho_{(1)}} = \frac{U_{(1)}}{U_{(2)}} , \quad (4.3.17)$$

one sees that the slope of the spectrum of shock-accelerated particles depends only on this quantity:

$$s = \frac{r + 2}{r - 1} . \quad (4.3.18)$$

This explains the attractiveness of this mechanism: it naturally produces a power-law momentum distribution with a slope that is determined by just a *single* parameter: the shock compression ratio. This is not the case in Fermi's original mechanism: there the slope depends on the ratio of the escape- and acceleration time $\tau_{\text{acc}}/T = \alpha T$, and there is no a-priori reason why this quantity takes on a value $\alpha T \sim 1$ in most sources so that $s = 1 + 1/\alpha T \sim 2$. Power-law distributions of energetic particles seem to be the norm in astrophysics: we already encountered an example of such a distribution in the spectrum of Galactic cosmic rays. The relativistic electrons responsible for the radio emission in supernova remnants or in the jets associated with some active galaxies also have such a power-law distribution over a wide range of energies.

In an ideal gas with specific heat ratio $\tilde{\gamma} = c_p/c_v = 5/3$, the shock compression ratio for very strong shocks has the maximum value⁸

$$r_{\max} = \frac{\tilde{\gamma} + 1}{\tilde{\gamma} - 1} = 4. \quad (4.3.19)$$

This corresponds to $s = 2$. This is the *minimum* value for s . In weaker shocks one has $r < r_{\max}$ and $s > 2$. It is therefore rather encouraging that the production spectrum of energetic electrons (or in some cases, protons) that one infers to be present in many astrophysical sources of HE particles/photons indeed follows a power-law, $\mathcal{N}(\mathcal{E}) \propto \mathcal{E}^{-s}$, with a slope $s \sim 2.2$ that is close to the minimum value

In reality, things are a bit more complicated than sketched here: first of all, the above calculation neglects all *energy losses*, which will modify the spectrum, usually at the highest energies where the losses are strongest. In particular, losses introduce a maximum energy $\mathcal{E}_{\max}$ beyond which the shock can not produce particles, as discussed in more detail below. Secondly, it assumes that the flow near the shock is not affected by the presence of the accelerated particles. The energetic particles (collectively as a gas) have pressure, and since their concentration increases towards the shock⁹, the associated pressure gradient will lead to a force acting on the incoming gas that points away from the shock. This pressure force will slow down the incoming flow, and weaken the strength of the shock. The theory of such *cosmic ray mediated shocks* is rather complicated. If shocks are very efficient accelerators that convert a significant fraction of the kinetic energy density in the incoming flow (equal to $\frac{1}{2}\rho_{(1)}U_{(1)}^2$) into the cosmic ray pressure, this modification of the incoming flow can be a significant effect.

⁸See for instance: L.D. Landau & E.M. Lifshitz 1959: *Fluid Mechanics*, Ch. 85, Pergamon Press, Oxford, UK.

⁹In a steady state the concentration of accelerated particles ahead of the shocks decays with distance x from the shock as $n(x) = n_0 \exp(-U_{(1)}x/\kappa_{\parallel(1)})$ as a result of the competition of diffusion of particles away from the shock with diffusion coefficient $\kappa_{\parallel(1)}$, and advection into the shock by the flow with velocity $U_{(1)}$. The concentration behind the shock is uniform in the steady state.

Energy gain, escape probability and power laws

The results for the momentum distribution from diffusive shock acceleration (which in the calculation presented here neglects the back-reaction of the accelerated particles on the flow) and the result (3.3.10) for the energy distribution from regular Fermi acceleration as discussed in Section 3.3 have a common origin. Both find a power-law in energy or momentum with a slope that is determined by the competition between the energy gain per unit time (or per shock crossing cycle) on the one hand, and the probability of escape on the other hand. There is a simple way of explicitly demonstrating this using a probabilistic approach.

Look at a number of discrete times $t_1, t_2, \dots, t_n, t_{n+1}, \dots$, and denote the energy at time t_n by $\mathcal{E}_n$. Let the energy after a time interval $\Delta t = t_{n+1} - t_n$ increase according to

$$\frac{\mathcal{E}_{n+1}}{\mathcal{E}_n} = 1 + \epsilon \quad (4.3.20)$$

with $\epsilon > 0$. Let particles escape from the accelerator so that their number satisfies

$$\frac{N_{n+1}}{N_n} = \mathcal{P}_{\text{surv}} = 1 - \mathcal{P}_{\text{esc}} . \quad (4.3.21)$$

Here $\mathcal{P}_{\text{surv}}$ ($\mathcal{P}_{\text{esc}}$) is the survival (escape) probability over a time Δt , and $N_n \equiv N(t_n)$. In our discussion of Shock Acceleration we called the survival probability $\mathcal{P}_{\text{ret}}$.

Assume that ϵ and $\mathcal{P}_{\text{esc}}$ do not depend on energy. In that case we have after a time $k \Delta t$, starting with N_0 particles in an energy interval $\mathcal{E}_0, \mathcal{E}_0 + d\mathcal{E}_0$ at $t = 0$:

$$N_k = (1 - \mathcal{P}_{\text{esc}})^k N_0 , \quad \mathcal{E}_k = (1 + \epsilon)^k \mathcal{E}_0 . \quad (4.3.22)$$

Taking the logarithm of both sides of these two relations gives:

$$\ln(\mathcal{E}_k/\mathcal{E}_0) = k \ln(1 + \epsilon) , \quad (4.3.23)$$

$$\ln(N_k/N_0) = k \ln(1 - \mathcal{P}_{\text{esc}}) .$$

Eliminating k in terms of $\ln(\mathcal{E}_k/\mathcal{E}_0)$ gives the number of particles N_k as a function of energy $\mathcal{E}_k$:

$$\ln(N_k/N_0) = \ln(\mathcal{E}_k/\mathcal{E}_0) \left[\frac{\ln(1 - \mathcal{P}_{\text{esc}})}{\ln(1 + \epsilon)} \right] . \quad (4.3.24)$$

This is equivalent with

$$N_k = N_0 \left(\frac{\mathcal{E}_k}{\mathcal{E}_0} \right)^{-\bar{s}}, \quad (4.3.25)$$

where

$$\bar{s} = \frac{\ln(1/(1 - \mathcal{P}_{\text{esc}}))}{\ln(1 + \epsilon)}. \quad (4.3.26)$$

Note that $\bar{s}$ is always positive if $\epsilon > 0$! Now assume that ϵ and $\mathcal{P}_{\text{esc}}$ are both small so that one can use:

$$\ln(1/(1 - \mathcal{P}_{\text{esc}})) \simeq \ln(1 + \mathcal{P}_{\text{esc}}) \simeq \mathcal{P}_{\text{esc}}, \quad \ln(1 + \epsilon) \simeq \epsilon. \quad (4.3.27)$$

In that case:

$$\bar{s} \simeq \frac{\mathcal{P}_{\text{esc}}}{\epsilon} = \frac{\mathcal{P}_{\text{esc}}}{\Delta\mathcal{E}/\mathcal{E}}. \quad (4.3.28)$$

Here I have used that $\epsilon = \Delta\mathcal{E}/\mathcal{E}$. Relation (4.3.25) gives the total number of particles left from the bunch that started in an energy interval $d\mathcal{E}_0$. If ϵ does not depend on energy the surviving particles are now in an energy interval $d\mathcal{E}$ that follows from:

$$d\mathcal{E}_k = \frac{d\mathcal{E}_k}{d\mathcal{E}_0} d\mathcal{E}_0 = (1 + \epsilon)^k d\mathcal{E}_0 = \frac{\mathcal{E}_k}{\mathcal{E}_0} d\mathcal{E}_0. \quad (4.3.29)$$

Since the energy distribution $\mathcal{N}(\mathcal{E})$ is defined (at any time or energy) by $dN = \mathcal{N}(\mathcal{E}) d\mathcal{E}$ it follows that $N_0 = \mathcal{N}(\mathcal{E}_0) d\mathcal{E}_0$ and $N_k = \mathcal{N}(\mathcal{E}_k) d\mathcal{E}_k$. Thus the energy distribution follows from (4.3.25) as:

$$\begin{aligned} \mathcal{N}(\mathcal{E}_k) &= \frac{N_k}{d\mathcal{E}_k} = \mathcal{N}(\mathcal{E}_0) \frac{d\mathcal{E}_0}{d\mathcal{E}_k} \left(\frac{\mathcal{E}_k}{\mathcal{E}_0} \right)^{-\bar{s}} \\ &= \mathcal{N}(\mathcal{E}_0) \left(\frac{\mathcal{E}_0}{\mathcal{E}_k} \right)^{-(1+\bar{s})}. \end{aligned} \quad (4.3.30)$$

The slope s of this distribution, when written in the standard form as $\mathcal{N}(\mathcal{E}) \propto \mathcal{E}^{-s}$, equals:

$$s = 1 + \bar{s} \simeq 1 + \frac{\mathcal{P}_{\text{esc}}}{\Delta\mathcal{E}/\mathcal{E}} . \quad (4.3.31)$$

This relation for the slope of the energy distribution is the main result of this approach.

It is easily demonstrated that we can recover the known results from this expression for s . In the simple model for regular Fermi acceleration discussed above in Chapter 3.3, where $d\mathcal{E}/dt = \alpha\mathcal{E}$ in an accelerator with escape time T , we have for sufficiently small Δt :

$$\frac{\mathcal{E}_{n+1}}{\mathcal{E}_n} = e^{\alpha\Delta t} \simeq 1 + \alpha\Delta t , \quad \frac{N_{n+1}}{N_n} = e^{-\Delta t/T} \simeq 1 - \frac{\Delta t}{T} , \quad (4.3.32)$$

which corresponds to $\epsilon = \alpha\Delta t$ and $\mathcal{P}_{\text{esc}} = \Delta t/T$. Relation (4.3.31) then gives:

$$s = 1 + \frac{1}{\alpha T} , \quad (4.3.33)$$

which (together with 4.3.30) corresponds to result (3.3.10).

In the case of diffusive shock acceleration the obvious choice for Δt is the cycle time, and one has for relativistic particles with $v = c$ and $\mathcal{E} \simeq pc$:

$$\epsilon = \frac{\Delta\mathcal{E}}{\mathcal{E}} = \frac{4}{3} \frac{U_{(1)} - U_{(2)}}{c} , \quad \mathcal{P}_{\text{esc}} = \frac{4U_{(2)}}{c} \quad (4.3.34)$$

and relation (4.3.31) immediately yields:

$$s = \frac{r+2}{r-1} \text{ with } r = U_{(1)}/U_{(2)} . \quad (4.3.35)$$

This is relation (4.3.18).

If one continuously injects fresh particles at energy $\mathcal{E}_0$, the energy distribution (4.3.30) will be established at *all* energies $\geq \mathcal{E}_0$ since particles of all ages are in principle present in the source.

4.3.2 Scattering, diffusion and the acceleration rate

The results derived so far have used a single assumption: the near-isotropy of the particle distribution near the shock. It is commonly assumed that this isotropy is the result of frequent scattering by low-frequency hydromagnetic waves, the same process responsible for the confinement of cosmic rays in the galaxy for some 10^7 years. Whenever cosmic rays try to stream with respect to a magnetized plasma with magnetic field B and density ρ , with a net transport velocity exceeding the Alfvén velocity $V_A = B/\sqrt{4\pi\rho}$, the cosmic rays will generate of Alfvén waves through a *gyroresonant instability*. The wavelength λ of these waves is comparable with the particle gyration radius: $\lambda \sim r_g(\mathcal{E}) \sim p_\perp c/qB_0$, with p the particle momentum, q the particle charge and B_0 the strength of the ambient magnetic field¹⁰.

The self-generated waves provide the magnetic fields which deflect the particles, and limit the mean free path of a charged particle along the field to a value

$$\ell_{\text{mfp}} \sim \frac{r_g(\mathcal{E})}{\mathcal{I}_{\text{magn}}(k \sim r_g(\mathcal{E})^{-1})} , \quad (4.3.36)$$

where $\mathcal{I}_{\text{magn}}(k)$ is the *relative* intensity of the magnetic fluctuations around a wavenumber $k = 2\pi/\lambda$ that is defined in such a way the rms amplitude B_{rms} of the fluctuating magnetic fields in a steady background field B_0 is given by

$$B_{\text{rms}}^2 = B_0^2 \int d \ln k \mathcal{I}_{\text{magn}}(k) . \quad (4.3.37)$$

This scattering leads to the *self-confinement* of cosmic rays, where the cosmic rays are coupled to the fluid motions by the hydromagnetic waves. As a result, they can not stream freely along the magnetic field. Instead, particles diffuse along the magnetic field. As we will see below, this diffusion along the field in the immediate vicinity of the shock determines the *acceleration rate*: the rate with which particles gain energy.

The gyroresonant scattering by low-frequency magnetic waves forces energetic charged particles to diffuse with respect to the flow. Standard diffusion theory says that a one-dimensional random walk along the magnetic field¹¹ has an associated diffusion coefficient

$$\kappa_{\parallel} = \frac{v \ell_{\text{mfp}}}{3} . \quad (4.3.38)$$

¹⁰For a review see: D.G. Wentzel, *Ann. Rev. Astron. Astrophys.* **12**, 71, 1974

¹¹There is also diffusion *across* the magnetic field. I will neglect this process for the moment.

If we orient the magnetic field along the z -axis ($\mathbf{B} = B \hat{z}$) and assume that the bulk flow of the plasma is along the magnetic field, this diffusion will lead to a transport equation *in space* of the form¹²:

$$\frac{\partial n(\mathbf{x}, t)}{\partial t} = -\frac{\partial}{\partial z} \left(V_{\parallel} n - \kappa_{\parallel} \frac{\partial n}{\partial z} \right) \quad (4.3.39)$$

Here $n(\mathbf{x}, t)$ is the number density of the diffusing particles, and $V_{\parallel}$ is the flow velocity of the plasma along the magnetic field. If the plasma velocity $\mathbf{V}$ has components perpendicular to $\mathbf{B}$, the flow term is replaced by $-\nabla \cdot (n \mathbf{V})$.

The fact that charged particles slowly diffuse with respect to the plasma allows one to estimate how long a particle remains upstream of the shock, if it is scattered back across the shock from the downstream side. As discussed in Section 2.5, a particle will -on average- travel a distance $\sim \ell_{\text{mfp}}$ before it starts to diffuse. This is a slow process, and at the same time the plasma is swept into the shock with velocity $U_{(1)}$. This means that a particle will spend a time of order

$$t_{\text{up}} \sim \ell_{\text{mfp}}/U_{(1)} \quad (4.3.40)$$

upstream before the shock sweeps across the particle and the particle (re-)enters the downstream flow. A more precise calculation¹³ yields:

$$t_{\text{up}} = \frac{4\ell_{\text{mfp}}}{3U_{(1)}}. \quad (4.3.41)$$

It turns out that the time that the particle resides in the downstream flow is given by the direct analogue of the upstream residence time:

$$t_{\text{down}} = \frac{4\ell_{\text{mfp}}}{3U_{(2)}}, \quad (4.3.42)$$

at least for those particles that do not escape, and are scattered back into the upstream flow. The cycle time for completing an full shock crossing cycle is therefore:

$$t_{\text{cy}} = t_{\text{up}} + t_{\text{down}} = \frac{4}{3} \left(\frac{(\ell_{\text{mfp}})_{(1)}}{U_{(1)}} + \frac{(\ell_{\text{mfp}})_{(2)}}{U_{(2)}} \right). \quad (4.3.43)$$

¹²In fact, the full theory combines the flow in configuration space and in momentum space to a *single* transport equation for the density of particles in phase space $(\mathbf{x}, \mathbf{p})$.

¹³L.O'C. Drury, *Rep. Prog. Phys.* **46**, 963, 1983.; see also the discussion in Section 2.4.1.

Here I have allowed for the possibility that the scattering mean free path ℓ_{mfp} takes different values in the up- and downstream flow.

We are now in a position to calculate the acceleration rate due to Diffusive Shock Acceleration. From the momentum gain (4.3.1) per cycle, and from expression (4.3.43) for the cycle time, one finds a mean momentum gain per unit time equal to:

$$\boxed{\left(\frac{dp}{dt}\right)_{\text{DSA}} = \frac{\Delta p}{t_{\text{cy}}} = \frac{1}{3} \left(\frac{U_{(1)} - U_{(2)}}{\frac{\kappa_{\parallel(1)}}{U_{(1)}} + \frac{\kappa_{\parallel(2)}}{U_{(2)}}} \right) p .} \quad (4.3.44)$$

Here I have used $v\ell_{\text{mfp}} = 3\kappa_{\parallel}$ to express the acceleration rate in terms of the diffusion coefficient.

It is often assumed that the magnetic fluctuations near a shock reach an amplitude comparable to that of the mean magnetic field: $|\delta\mathbf{B}| \sim B_0$ (or equivalently: $\mathcal{I}_{\text{magn}} \simeq 1$). In that case, the above expression for the scattering mean free path ℓ_{mfp} gives $\ell_{\text{mfp}} \approx r_g \sim pc/qB$, and the diffusion proceeds at the so-called *Bohm rate*, with a diffusion coefficient equal to:

$$\kappa_{\parallel} \approx \frac{vr_g}{3} \sim \frac{pcv}{3qB} \equiv \kappa_B . \quad (4.3.45)$$

For the rest of this Section, I will limit myself to relativistic particles with $\gamma \gg 1$ so that $v \sim c$ and $\mathcal{E} \sim pc$. Using $U_{(1)} = V_s$ (the shock velocity), $U_{(2)} = U_{(1)}/4$ (strong non-relativistic shock) and assuming (for simplicity) that $B_{(1)} = B_{(2)} \equiv B$, the acceleration rate for $\kappa_{\parallel} \approx \kappa_B$ is:

$$\left(\frac{d\mathcal{E}}{dt}\right)_B \approx \frac{3}{20} \frac{V_s^2}{c} qB . \quad (4.3.46)$$

This expression lends itself for a near-perfect illustration of our formula for $\mathcal{E}_{\text{max}}$, Eqn. (2.2.3). The above calculation is for non-relativistic shocks with $V_s \ll c$ so $\Gamma_s \approx 1$.

If there are no energy losses, and the shock is an expanding blast wave of size R_s , it's age must be of order¹⁴ $t_s \sim R_s/V_s$. Relation (4.3.46) predicts that the typical energy that can be attained in a time t_s is

$$\mathcal{E} \sim \left(\frac{d\mathcal{E}}{dt} \right)_B t_s = \frac{3}{20} \frac{V_s^2 t_s}{c} qB \sim \frac{3}{20} q\beta_s R_s B, \quad (4.3.47)$$

with $\beta_s = V_s/c$.

4.4 Effect of energy losses

The simple power law $\mathcal{N}(p) dp \propto p^{-s} dp$ with $s = (r+2)/(r-1)$ predicted for particles accelerated at a shock was derived neglecting energy losses. Usually, these losses increase with increasing particle energy, for instance: the synchrotron loss rate of relativistic electrons scales as $(d\mathcal{E}/dt)_{\text{sy}} \propto \mathcal{E}^2$. If the losses incurred in one shock crossing cycle exceed the gain due to shock acceleration, the spectrum must cut off. Therefore, the power-law extends to an energy $\mathcal{E}_{\text{max}} = \sqrt{p_{\text{max}}^2 c^2 + m^2 c^4}$ that is determined by the condition that the momentum gains balance the losses¹⁵, so that at $p = p_{\text{max}}$ one has:

$$\Delta p \approx \frac{4}{3} \frac{U_{(1)} - U_{(2)}}{v} p = \int_0^{t_{\text{cy}}} dt \left(\frac{dp}{dt} \right)_{\text{loss}} \approx t_{\text{cy}} \left(\frac{dp}{dt} \right)_{\text{loss}}. \quad (4.4.1)$$

I will limit the rest of the discussion to very relativistic particles for which the energy-momentum relation can be approximated by $\mathcal{E} \approx pc$ and the velocity particle satisfies $v \approx c$. Assuming a maximally efficient shock, the acceleration rate due to DSA can be written as:

$$\frac{d\mathcal{E}}{dt} = \xi qBc \beta_s^2, \quad (4.4.2)$$

where $\xi = 3/20$ if particles diffuse at the Bohm rate and the magnetic field is along the shock normal, and smaller if the diffusion is slower (when $|\delta \mathbf{B}| < B_0$ so that $\ell_{\text{mfp}} > r_g$).

¹⁴In fact, the well-known Sedov-Taylor expansion law for a spherical remnant expanding into a uniform medium at rest gives $R_s \propto t^{2/5}$ and $t_s = \frac{2}{5} (R_s/V_s)$

¹⁵I use the convention that the momentum change due to losses is written with a minus sign as $dp/dt = -(dp/dt)_{\text{loss}}$ so that $(dp/dt)_{\text{loss}} > 0$.

I will define an *acceleration rate* $\mathcal{R}_{\text{acc}}$ and a *rate* $\mathcal{R}_{\text{loss}}$ by

$$\mathcal{R}_{\text{acc}} \equiv \frac{1}{\mathcal{E}} \frac{d\mathcal{E}}{dt} = \frac{\xi q B c \beta_s^2}{\mathcal{E}} , \quad \mathcal{R}_{\text{loss}} \equiv \frac{1}{\mathcal{E}} \left(\frac{d\mathcal{E}}{dt} \right)_{\text{loss}} . \quad (4.4.3)$$

Both rates usually are functions of the energy $\mathcal{E}$.

The maximum energy that can be produced in an acceleration process follows from the balance between gains and losses at that energy:

$$\mathcal{R}_{\text{acc}}(\mathcal{E}_{\text{max}}) = \mathcal{R}_{\text{loss}}(\mathcal{E}_{\text{max}}) . \quad (4.4.4)$$

The following losses occur most frequently in astrophysically relevant situations:

- **Adiabatic losses.** In expanding and almost spherical shock waves, such as those found in supernova remnants, the wave-mediated coupling between the accelerated particles and the expanding downstream gas leads to expansion losses, with a loss rate proportional to the divergence of the flow speed $\mathbf{V}$:

$$\mathcal{R}_{\text{loss}}^{\text{exp}} = \frac{1}{3} \nabla \cdot \mathbf{V} . \quad (4.4.5)$$

In supernova remnants older than ~ 1000 years the expansion follows the well-known Sedov-Taylor law, $R_s(t) \propto t^{2/5}$ for a spherical remnant expanding into a uniform interstellar medium. The Sedov solution for the flow velocity $V(r)$ as a function of radius r inside the remnant gives, directly behind the blast wave for $R_s - r \ll R_s$,

$$\frac{V(r)}{V_s} \approx \frac{3}{4} \left(\frac{2r}{R_s} - 1 \right) \longleftrightarrow \nabla \cdot \mathbf{V} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V(r)) \approx \frac{3V_s}{R_s} . \quad (4.4.6)$$

Here R_s is the shock radius and V_s the shock speed. The typical loss rate is $\mathcal{R}_{\text{loss}}^{\text{exp}} \sim V_s/R_s$, and criterion (4.4.4) yields:

$$\mathcal{E}_{\text{max}} \sim |Z| e B \xi \beta_s R_s , \quad (4.4.7)$$

with $\beta_s \equiv V_s/c$. A similar value for $\mathcal{E}_{\text{max}}$ is obtained from if one assumes that the finite age of the source, $t_s \sim 2R_s/5V_s$ determines the maximum attainable energy.

- **Synchrotron/inverse Compton losses.** For relativistic electrons, and or for ultra-relativistic protons, emission of synchrotron radiation and inverse Compton radiation are an important loss mechanism. In the Thomson limit¹⁶, and for relativistic particles ($v \sim c$, $\mathcal{E} \sim pc$), these losses increase with energy so that the loss rate scales as:

$$\mathcal{R}_{\text{loss}}^{\text{sc}}(\mathcal{E}) = \frac{1}{\tau_{\text{loss}}} = \frac{1}{t_{\text{sc}}} \left(\frac{\mathcal{E}}{mc^2} \right). \quad (4.4.8)$$

The time t_{sc} appearing in this expression is given by

$$t_{\text{sc}} = \begin{cases} \frac{6\pi m_e c}{\sigma_T B_e^2} \equiv t_{\text{sc}}^e & \text{electrons;} \\ \left(\frac{m_p}{m_e} \right)^3 \left(\frac{A^3}{Z^4} \right) t_{\text{sc}}^e & \text{nuclei with mass } m = A m_p \text{ and charge } q = Ze. \end{cases} \quad (4.4.9)$$

Note that the loss time at an energy $\mathcal{E}$ is $\tau_{\text{loss}} = t_{\text{sc}}(\mathcal{E}/mc^2)^{-1} = t_{\text{sc}}/\gamma$, with γ the Lorentz factor of the energetic charge. In this expression $B_e \equiv \sqrt{B^2 + 8\pi U_{\text{rad}}}$ is an effective magnetic field, which includes the effect of the radiation field (and the associated inverse Compton losses) through the inclusion of the ambient radiation density U_{rad} .

If synchrotron/inverse-Compton losses are the factor determining the maximum energy one has

$$\mathcal{E}_{\text{max}} \sim \sqrt{mc^2} \beta_s (ZeB \xi \ell_{\text{sc}})^{1/2}, \quad (4.4.10)$$

with $\ell_{\text{sc}} \equiv ct_{\text{sc}}$. Note that this energy scales as $\mathcal{E}_{\text{max}} \propto B^{-1/2}$, since $\ell_{\text{sc}} \propto B^{-2}$.

- **Pion production losses.** Highly-relativistic protons and nuclei moving through a non-relativistic ambient medium, or through an intense radiation field, suffer collisions with nuclei (N) or photons (γ). Above a certain threshold energy, these collisions can lead to pion production in reactions like $p + \gamma \longrightarrow p + \pi$ and $p + N \longrightarrow p + N + \pi$. These processes can become an important loss mechanism, in particular in Active Galactic Nuclei where the radiation density is very high.

¹⁶This is the limit where the relevant cross-section for Compton scattering by electrons is the Thomson cross section $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$.

Pion production in collisions with photons is the most important mechanism for limiting the energy in the possible production sites of the *Ultra High Energy Cosmic Rays* (UHECRs), such as the jets of Active Galaxies or the fireballs associated with Gamma Ray Bursts, provided such particles are indeed the result of acceleration from lower energies as opposed to -say- the product of some exotic process such as the decay of a hypothetical supermassive particle.

Typically, the pion losses can be characterized by a *loss length* that can be written as:

$$\ell_\pi = \frac{1}{n_t \sigma_\pi K_p} . \quad (4.4.11)$$

Here n_t is the number of the target particles in the pion-producing collision (photons or nuclei), σ_π is the collision cross section, and K_p is the *inelasticity parameter*: $K_p = |\Delta\mathcal{E}|/\mathcal{E}$ is the fraction of the energy lost per collision. Typically, it takes a value $K_p \approx 0.2 - 0.5$. The relevant cross sections for pion production in collisions with photons and nucleons are for protons:

$$\sigma_{p\gamma} \simeq 10^{-28} \text{ cm}^2 ;$$

$$\sigma_{pN} \simeq 5 \times 10^{-26} \text{ cm}^2 .$$

In the simplest case of a constant loss length ℓ_π , so that the losses are

$$\mathcal{R}_{\text{loss}}^\pi = c/\ell_\pi , \quad (4.4.12)$$

the maximum energy attainable in non-relativistic shocks equals

$$\mathcal{E}_{\text{max}} \sim |Z|eB \xi \beta_s^2 \ell_\pi . \quad (4.4.13)$$

The production of pions always has a threshold: the incoming proton has to have a minimum energy $\mathcal{E}_{\text{min}}$ for the reaction to take place. As an example I will consider pion production on photons. Near threshold, this reaction actually proceeds via the production of an intermediary particle, the $\Delta^+(1232)$, a positively charged short-lived particle (resonance) consisting of three quarks (just like the proton is a three-quark composite) with a mass of 1232 MeV.

The reaction involved looks like:

$$p + \gamma \implies \Delta^+(1232) \implies \begin{cases} p' + \pi^0 & \text{neutral pion production;} \\ n + \pi^+ & \text{charged pion production.} \end{cases} \quad (4.4.14)$$

Let the incoming proton have a momentum four-vector $P^\mu = (\mathcal{E}, \mathbf{p})$, with $\mathcal{E}^2 - |\mathbf{p}|^2 = m_p^2$ with $m_p \simeq 938$ MeV the proton mass, and let the photon (remember: the photon mass is zero!) have a momentum four vector equal to $q^\mu = (\epsilon, \mathbf{q})$, with $\epsilon = |\mathbf{q}|$, so that $\epsilon^2 - |\mathbf{q}|^2 = 0$. The four-momentum of the outgoing proton or neutron is $P'^\mu \equiv (\mathcal{E}', \mathbf{p}')$, and the pion that is produced in this reaction has a four-momentum $Q^\mu = (\mathcal{E}_\pi, \mathbf{Q})$, where $\mathcal{E}_\pi^2 - |\mathbf{Q}|^2 = m_\pi^2$, with $m_\pi \simeq 135$ MeV the pion mass.

In this reaction, the total energy and total momentum is conserved. This is most conveniently expressed using energy-momentum four vectors. The momentum four-vector of a particle with energy $\mathcal{E}$, momentum $\mathbf{p}$ and rest mass m is, in a notation where $c = 1$:

$$P^\mu = (\mathcal{E}, \mathbf{p}) . \quad (4.4.15)$$

The Lorentz-invariant length squared of this four-vector yields the relation

$$P^2 \equiv \mathcal{E}^2 - |\mathbf{p}|^2 = m^2 . \quad (4.4.16)$$

In what follows, I will use this particular relation (and others like it) again and again.

In the language of four-vectors energy-momentum conservation in reaction can be expressed by the single equation

$$P^\mu + q^\mu = P_\Delta^\mu = P'^\mu + Q^\mu \equiv s^\mu . \quad (4.4.17)$$

Here P_Δ^μ is the four-momentum of the $\Delta(1232)$ -resonance. Since the total four-momentum before and after the reaction is the same, one can define a Lorentz-invariant conserved quantity, the *total four-momentum squared*:

$$s^2 \equiv (P + q)^2 \quad (4.4.18)$$

Momentum and energy conservation together imply that s^2 does *not* change in the reaction: $s_{\text{after}}^2 = (P' + Q)^2 = s_{\text{before}}^2$. In the case where the Δ -resonance dominates the production process one has:

$$s^2 = P_{\Delta}^2 = m_{\Delta}^2 . \quad (4.4.19)$$

The scalar product of two four-vectors $A^{\mu} = (A^0, \mathbf{A})$ and $B^{\mu} = (B^0, \mathbf{B})$ is defined as¹⁷

$$A \cdot B = A^0 B^0 - \mathbf{A} \cdot \mathbf{B} . \quad (4.4.20)$$

Using this definition, it is easy to show that the invariant quantity s before the reaction equals

$$\begin{aligned} s_{\text{before}}^2 &= (P + q)^2 = P^2 + q^2 + 2P \cdot q \\ &= m_p^2 + 2(\mathcal{E}\epsilon - \mathbf{p} \cdot \mathbf{q}) . \end{aligned} \quad (4.4.21)$$

Here I have used that photons are mass-less so that $q^2 = 0$. For the reaction to proceed, one needs to be able to excite the Δ -resonance, so the threshold condition (approximately) reads:

$$s_{\text{before}}^2 \geq m_{\Delta}^2 , \quad (4.4.22)$$

or equivalently:

$$\mathcal{E}\epsilon - \mathbf{p} \cdot \mathbf{q} \geq \frac{1}{2} (m_{\Delta}^2 - m_p^2) . \quad (4.4.23)$$

For ultra-relativistic protons with $|\mathbf{p}| \simeq \mathcal{E}$ one has $\mathbf{p} \cdot \mathbf{q} \simeq \mathcal{E} \epsilon \cos \theta$ with θ the angle between the momenta of the incoming proton and the incoming photon.

¹⁷See for instance: L.D. Landau & E.M. Lifshitz 1975: *Classical Theory of Fields*, 4th Ed., Ch. 1, Pergamon Press, Oxford, UK.

It is easily checked that the threshold condition is most easily met in a head-on collision between proton and foton where $\cos\theta = -1$. That determines the minimum proton energy (given the photon energy ϵ) for the Δ -resonance as:

$$\mathcal{E}_{\min} \simeq \frac{m_{\Delta}^2 - m_p^2}{4\epsilon} = 1.59 \times 10^{17} \left(\frac{\epsilon}{1 \text{ eV}} \right)^{-1} \text{ eV} . \quad (4.4.24)$$

If one ‘forgets’ about the Δ -resonance, and just looks at the final products of the reaction, one finds a slightly less stringent threshold condition¹⁸:

$$\mathcal{E}_{\min} = \frac{m_{\pi} m_p (1 + m_{\pi}/2m_p)}{2\epsilon} = 6.79 \times 10^{16} \left(\frac{\epsilon}{1 \text{ eV}} \right)^{-1} \text{ eV} . \quad (4.4.25)$$

This last calculation assumes that (at threshold) the outgoing proton (or neutron) and pion are produced at rest in the *center of momentum frame* (CMF). This special frame is defined by the following relation on the three-momenta:

$$(\mathbf{p} + \mathbf{q})_{\text{CMF}} = (\mathbf{p}' + \mathbf{Q})_{\text{CMF}} = \mathbf{0} . \quad (4.4.26)$$

Momentum conservation ensures that the center of momentum frame is the same before and after the reaction. This condition also reveals an important relation between the quantity s and the *total* conserved energy $\mathcal{E}_{\text{CMF}}$ in the CMF:

$$s^2 = \mathcal{E}_{\text{CMF}}^2 . \quad (4.4.27)$$

In practice, it does not matter much which of these two thresholds one uses: the Δ -resonance is not sharp. It has a width of about 120 MeV so that photo-pion production is already possible for slightly lower energies than the threshold energy of Eqn. (4.4.24). Well above the threshold energy the pion production mechanism changes qualitatively, and no longer involves the Δ resonance.

¹⁸See for instance: D. Perkins 2003: *Particle Astrophysics*, Ch. 6.6, Oxford University Press, Oxford, UK.

Table 2 on the next page gives the typical values of the maximum particle energy $\mathcal{E}_{\max}$ in a number of cosmic accelerators.

Table 2:
Maximum observer's frame energy in shock acceleration at the Bohm rate

Object	R_s	$\beta_s = V_s/c$ Γ_s	B (G)	limiting mechanism	$\mathcal{E}_{\max}$ (in eV)
Solar Flare Type II shock	10^{10} cm	$10^{-2.5}$ 1	10	<i>protons</i> : size/age <i>electrons</i> : synchr. losses	10^{11} 10^9
Interplanetary Shocks	10^{13} cm	10^{-3} 1	10^{-5}	<i>protons</i> : size/age <i>electrons</i> : size/age	10^8 10^8
Supernova Remnants	10 pc	$10^{-2.5}$ 1	10^{-4}	<i>protons</i> : size/age <i>electrons</i> : synchr. losses	10^{15} 10^{12}
Superbubbles	5 kpc	10^{-4} 1	10^{-4}	<i>protons</i> : size/age <i>electrons</i> : synchr. losses	$10^{15.5}$ 10^{11}
Gamma Ray Burst, external shock:	0.01 pc	1 $10^2 - 10^3$	10^{-6}	<i>protons</i> : size/age <i>electrons</i> : size/age	10^{15} 10^{15}
Gamma Ray Burst, internal shocks:	0.01 pc	~ 1 ~ 1	10^2	<i>protons</i> : size/age <i>electrons</i> : synchr. losses	$10^{20.5}$ $10^{15.5}$
AGN	10^{13} cm	0.1 1	10^4 (?)	<i>protons</i> : pion prod. losses <i>electrons</i> : SC losses	10^{16} 10^{11}
Hot Spots radiogalaxy	10 kpc	~ 1 1-10	10^{-4}	<i>protons</i> : shock size <i>electrons</i> : synchr. losses	10^{20} 10^{14}
Cosmic Large- Scale Structure	10 Mpc	$10^{-2.5}$ 1	10^{-6}	<i>protons</i> : pion prod. losses <i>electrons</i> : SC losses	10^{19} 10^{14}

Note: In the case of particles produced by internal shocks in the fireballs of Gamma Ray Bursts, the energies given in this Table are the energy as measured by an outside observer. Because of the relativistic bulk motion towards the observer in these objects with Lorentz factor Γ_s , this energy is $\sim \Gamma_s \times$ the energy in the rest-frame of the material. Taking $\Gamma_s \sim 10^3$, this means that the internal shocks can produce protons up to $\sim 10^{17.5}$ eV (and electrons up to $\sim 10^{12.5}$ eV) in the rest frame of the fireball, provided the magnetic energy density is in equipartition with the kinetic energy density so that $B \sim 100$ G (see below). The Lorentz-factor Γ , 'velocity' β and magnetic field strength B on the other hand all refer to the values measured in the frame comoving with the fireball material.

Intermezzo: gyro-resonant scattering of charged particles

The fact that galactic cosmic rays with $\mathcal{E} \sim 10$ GeV are confined to galaxies for $\sim 10^7$ years implies that they are undergoing a random walk with a stepsize of ~ 1 pc, and that something is scattering them. One can demonstrate that ordinary interactions between cosmic rays and the ions in the interstellar gas are totally ineffective in scattering the cosmic rays. We must seek an alternative scattering mechanism that does not rely on actual collisions between particles.

It is believed that this scattering mechanism is provided by scattering on the irregularities ('wrinkles') in the galactic magnetic field. This is a *collective process* in the sense that the cosmic rays are not scattered by the electromagnetic field of a single particle in the interstellar medium, but by a wave supported by the plasma as a whole! The same scattering mechanism is thought to be responsible for the confinement of charged particles near astrophysical shocks in *Diffusive Shock Acceleration*.

The main effect of this magnetic field on the charged cosmic rays is the magnetic component of the Lorentz force,

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_L = q \left(\frac{\mathbf{v}}{c} \times \mathbf{B} \right). \quad (4.4.28)$$

This forces cosmic rays (or any charged particle) to gyrate around the magnetic field with the gyration frequency Ω_g and gyration radius r_g . For a positive charge ($q > 0$) these are given by

$$\Omega_g = \frac{qB}{\gamma mc}, \quad r_g = \frac{v_\perp}{\Omega_g} = \frac{p_\perp c}{qB}. \quad (4.4.29)$$

Here the subscript $\perp$ denotes the component perpendicular to the magnetic field. If one takes the magnetic field to be uniform, and along the z -axis,

$$\mathbf{B}_0 = B_0 \hat{e}_z, \quad (4.4.30)$$

the equation of motion yields a helical orbit around the field with constant pitch:

$$\begin{aligned} \mathbf{p}(t) &= \left(p_\perp \cos(\Omega_g t + \alpha), -p_\perp \sin(\Omega_g t + \alpha), p_\parallel \right) \\ \mathbf{x}(t) &= \left(r_g \sin(\Omega_g t + \alpha), r_g \cos(\Omega_g t + \alpha), z_0 + v_\parallel t \right) \end{aligned} \quad (4.4.31)$$

Here $p_{\perp} = \sqrt{p_x^2 + p_y^2}$ and $p_{\parallel} = p_z$. For negative charges ($q < 0$) we have $r_g = cp_{\perp}/|q|B$ and $\Omega_g < 0$. In that case $\mathbf{x}(t) = (-r_g \sin(\Omega_g t + \alpha), -r_g \cos(\Omega_g t + \alpha), z_0 + v_{\parallel} t)$.

The Lorentz force leaves the component of momentum along the field unaffected since there is no force component along the field:

$$p_{\parallel} = \text{constant} . \quad (4.4.32)$$

The (magnetic) Lorentz force also does no work, i.e. the particle energy is conserved:

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt} \left(\sqrt{m^2 c^4 + p^2 c^2} \right) = \mathbf{F}_L \cdot \mathbf{v} = 0 . \quad (4.4.33)$$

This in turn implies that the *magnitude* of the perpendicular momentum must also remain constant:

$$p_{\perp} = \text{constant} . \quad (4.4.34)$$

The pitch angle θ of this orbit is defined by

$$\tan \theta = \frac{p_{\perp}}{p_{\parallel}} , \quad (4.4.35)$$

which remains constant as both $p_{\parallel}$ and $p_{\perp}$ are conserved. In a certain sense the cosmic rays 'slide' along the magnetic field as beads along a wire.

Let us now introduce a small random perturbation $\delta \mathbf{B} \perp \mathbf{B}_0$ in the magnetic field. Such a perturbation is typical for low-frequency waves propagating along the ambient field $\mathbf{B}_0$. 'Random' in this context means that a suitably defined average of $\delta \mathbf{B}$ (the ensemble average, see below) vanishes:

$$\langle \delta \mathbf{B} \rangle = 0 . \quad (4.4.36)$$

In this particular case I will assume that $\delta \mathbf{B}$ can be written as

$$\delta \mathbf{B} = \delta B [\cos \omega(t), \sin \omega(t), 0] , \quad (4.4.37)$$

where the angle $\omega(t)$ is randomly chosen at each wave-particle interaction, and lies in the range $0 \leq \omega \leq 2\pi$.

The associated random Lorentz force,

$$\delta \mathbf{F}_L = q \left(\frac{\mathbf{v}}{c} \times \delta \mathbf{B} \right) , \quad (4.4.38)$$

will change the pitch angle of the cosmic ray momentum $\mathbf{p}$ with respect to $\mathbf{B}_0$, while still leaving the particle energy (and $p = |\mathbf{p}|$) invariant. The typical change in parallel momentum induced by the perturbation $\delta \mathbf{B}$ is

$$\delta p_{\parallel} \simeq \frac{qv_{\perp} \delta B}{c} \sin \psi \Delta t_c , \quad (4.4.39)$$

with Δt_c the interaction time between wave and particle, and $\psi = \Omega_g(t) + \alpha - \omega$ the relative angle between $\mathbf{v}_{\perp}$ and $\delta \mathbf{B}$. The change $\delta \theta$ in the pitch-angle θ that is associated with $\delta p_{\parallel}$ follows from

$$\delta p_{\parallel} = -p \sin \theta \delta \theta , \quad v_{\perp} = v \sin \theta , \quad (4.4.40)$$

and equals

$$\delta \theta \simeq \frac{q \delta B}{\gamma m c} \sin \psi \Delta t_c . \quad (4.4.41)$$

This interaction only happens with waves with a wavelength of order $\lambda \simeq r_g$, and the typical interaction time equals

$$\Delta t_c \simeq \frac{2\pi}{\Omega_g} , \quad (4.4.42)$$

which is the gyration period of the particle around the magnetic field $\mathbf{B}_0$. This can be understood as follows, see the Figure below: an Alfvén wave propagating along the magnetic field with a wavelength λ have a magnetic field perturbation $\delta \mathbf{B}$ that rotates once around the field each wavelength: they are circularly polarized waves. Particles interact only effectively with these waves if the rotation associated with their gyration around the field is synchronous with the rotation of $\delta \mathbf{B}$. In that case, the angle between the particle momentum vector and the magnetic field perturbation remains the same. If the particles slide along the magnetic field with velocity $v_{\parallel}$, this means that they must travel one wavelength in one gyro-period:

$$v_{\parallel} \Delta t = \frac{2\pi v_{\parallel}}{\Omega_g} = \pm \lambda = \pm \frac{2\pi}{k} . \quad (4.4.43)$$

The choice of sign depends on charge of the particle and the rotation sense of the magnetic field. Here $k = 2\pi/\lambda$ is the wavenumber of the magnetic perturbation. This leads to the *resonance condition*

$$kv_{\parallel} \pm \Omega_g = 0, \quad (4.4.44)$$

and the resonant wavelength is $\lambda \approx v_{\parallel}/\Omega_g \sim r_g(\mathcal{E})$.

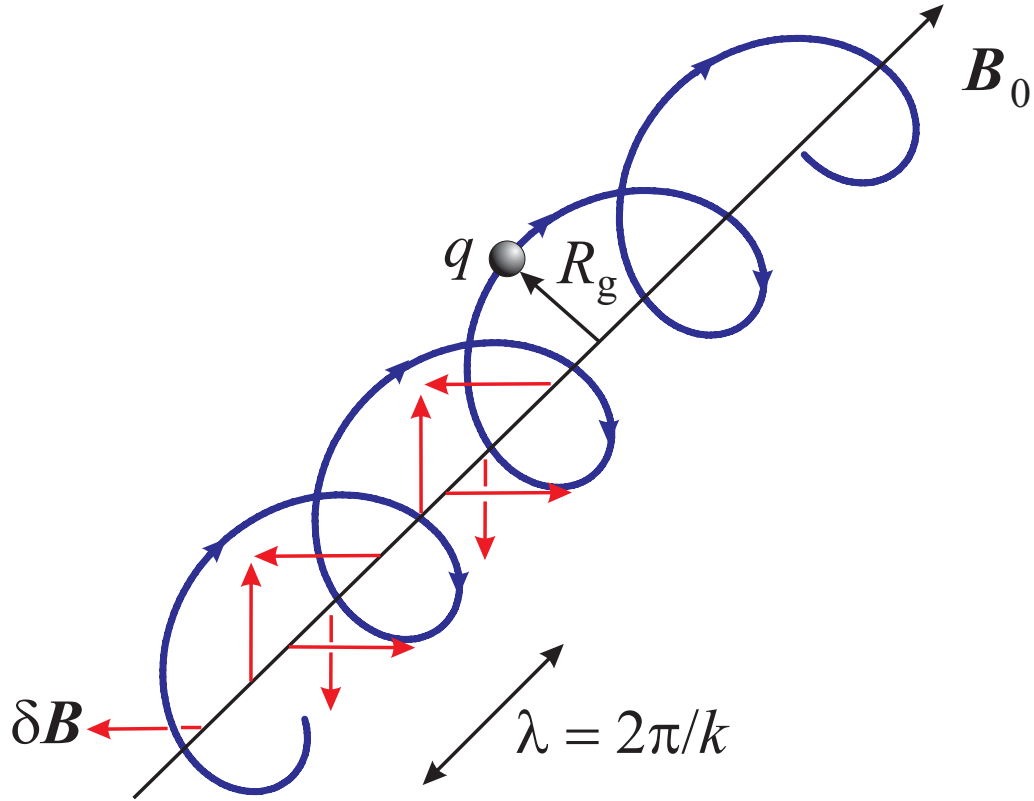


Figure 4.4: A charge q spirals in a helical orbit (shown in blue) around the mean magnetic field B_0 . The radius of gyration is r_g . An Alfvén wave with a magnetic perturbation $\delta B \propto B_0$ also propagates along the field. The vector δB (the red arrows) rotates once around B_0 each wavelength λ . A gyro-resonant particle, which is the case illustrated here, completes exactly one rotation around B_0 in a distance λ , so that the relative orientation of the magnetic perturbation vector and the velocity vector of the particle does not change.

This simple calculation neglects that the waves have a phase speed $v_{\text{ph}} = \omega/k$, with ω the wave frequency. If one takes this phase velocity into account the resonance condition becomes $\omega - kv_{\parallel} \pm \Omega_g = k(v_{\text{ph}} - v_{\parallel}) \pm \Omega_g = 0$.

This gives the important result that the change of the pitch angle per interaction with a *single* wave equals

$$\delta\theta(\psi) \simeq 2\pi \left(\frac{\delta B}{B_0} \right) \sin \psi . \quad (4.4.45)$$

If the waves are truly random, one can assume that the phase angle ψ is randomly chosen from a uniform distribution $0 \leq \psi \leq 2\pi$ at each interaction. This is called the *random phase approximation*. In that case, the mean value of the pitch angle change, averaged over many independent interactions, vanishes:

$$\langle \delta\theta \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} d\psi \delta\theta(\psi) = 0 . \quad (4.4.46)$$

This does not mean that the pitch angle does not change! Rather, the value of θ exhibits a *random walk*¹⁹. This means that after N interactions the net change in pitch angle equals

$$\Delta\theta \approx \sqrt{N} \delta\theta_{\text{rms}} . \quad (4.4.47)$$

Here $\delta\theta_{\text{rms}}$ is the root mean square value of the pitch angle change, which follows from:

$$\delta\theta_{\text{rms}}^2 \equiv \frac{1}{2\pi} \int_0^{2\pi} d\psi \delta\theta^2(\psi) . \quad (4.4.48)$$

Using expression (4.4.45) for $\delta\theta$, one finds:

$$\delta\theta_{\text{rms}}^2 = 2\pi^2 \left(\frac{\delta B}{B_0} \right)^2 . \quad (4.4.49)$$

The mean free path associated pitch angle scattering follows from the requirement $\Delta\theta \approx \pi$: the particle completely reverses its direction of motion along the magnetic field.

¹⁹For a readable introduction to such processes see: Ya. B. Zeldovich, A.A. Ruzmaikin & D.D. Sokoloff, *The Almighty Chance*, World Scientific Lecture Notes in Physics, Vol. 20, 1990.

The number of scattering needed for this follows from (4.4.47), and equals:

$$N_{\text{sc}} \sim \left(\frac{\pi}{\delta\theta_{\text{rms}}} \right)^2 \approx \frac{1}{4} \left(\frac{\delta B}{B_0} \right)^{-2}. \quad (4.4.50)$$

A particle travels a distance $\ell \approx v_{\parallel} \Delta t_c \approx 2\pi v_{\parallel} / \Omega_g$ between two subsequent interactions. Therefore, it will reverse its direction of motion completely after a distance $\ell_{\text{mfp}} = N_{\text{sc}} \ell$, which equals, up to factor of order unity:

$$\ell_{\text{mfp}} \sim \frac{\pi}{4} \frac{v}{\Omega_g} \left(\frac{\delta B}{B_0} \right)^{-2} \sim r_g \left(\frac{\delta B}{B_0} \right)^{-2}. \quad (4.4.51)$$

This is a simple derivation of Eqn. (4.3.36).
