Radiation Field - Key Concepts

In this section only some key issues will be highlighted. For a detailed discussion see Rybicki & Lightman (1979) or Judith Irwin's "Decoding the Cosmos".

Definitions:

Radiative Flux:

$$F = \frac{dE}{dA\,dt} \quad (\text{erg cm}^{-2}\,\text{s}^{-1})$$

The flux at a distance from an isotropic emitter

$$F = \frac{dE/dt}{dA}$$

$$= \frac{L_*}{4\pi r^2}$$

$$F \propto \frac{1}{r^2}$$

Intensity or integrated Brightness:

The brightness of a source of radiation is

$$I = \frac{dE}{dA\,dt\,d\Omega} \quad (\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{sr}^{-1})$$

Hence one can see that there is in general a connection between F and I.

From

$$I = \frac{dG}{dA\,dt\,d\Omega}$$

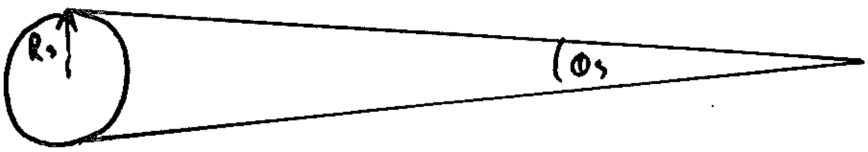
$$\Rightarrow \quad I\,\Delta\Omega = \frac{dG}{dA\,dt} = F$$

So one can say that the flux is the brightness beamed into a certain solid angle.

$$\boxed{F = I\,\Delta\Omega} \qquad \Delta\Omega = \frac{\Delta A}{r^2}$$

Two examples:
Determine the flux measured from:
a.) a spatially resolved source of radius R_s at distance r
b.) a point source at distance r if the telescope has radius R_{det}.
Remark: In both cases the brightness I_* is uniform

a.) Spatially Resolved Source



$$F = I_*\,\Delta\Omega$$

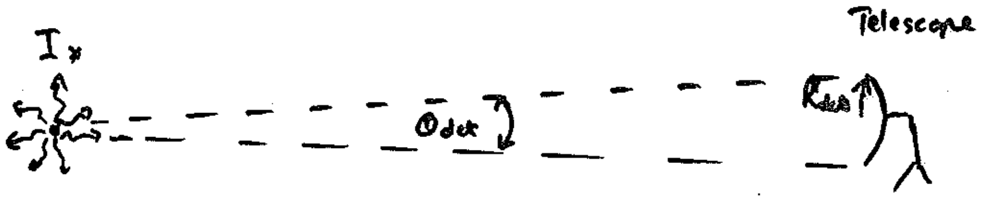
$$= I_*\,\frac{\Delta A}{r^2}$$

$$= I_* \times \frac{\pi R_s^2}{r^2}$$

$$= I_* \cdot \frac{\pi\left(\frac{d_s}{2}\right)^2}{r^2} \qquad \theta_s \approx \left(\frac{d}{r}\right)$$

$$= \frac{\pi}{4}\,I_*\left(\frac{d_s}{r}\right)^2 = \frac{\pi}{4}\,I_*\,\theta_s^2$$

b.) Point source



$$F = I_* \, \Delta\Omega$$
$$= I_* \times \frac{\pi R_{det}^2}{r^2}$$
$$= I_* \times \frac{\pi \left(\frac{D_{det}}{2}\right)^2}{r^2}$$
$$= \frac{\pi}{4} I_* \times \left(\frac{D_{det}}{r}\right)^2$$
$$= \frac{\pi}{4} I_* \times \Theta_{det}^2 .$$

One can see that to obtain measurable fluxes from point sources we want to build our telescopes as large as possible.

Since the radiation field from astrophysical sources contain spectral information that is associated with the physical process producing the radiation, we measure the flux in specific spectral bands

$$F_\nu = \frac{dF}{d\nu} \quad (\mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}})$$
$$= \frac{I \, \Delta\Omega}{d\nu}$$
$$= I_\nu \, \Delta\Omega \qquad \boxed{I_\nu = \frac{I}{d\nu}}$$

In the expression on the previous slide

$$I_\nu = \frac{dG}{dA\,dt\,d\Omega\,d\nu}$$ [specific intensity or Brightness]

$$\boxed{I_\nu \Rightarrow \text{erg cm}^{-2}\,\text{s}^{-1}\,\text{sr}^{-1}\,\text{Hz}^{-1}}$$

$\Rightarrow \quad F = \int F_\nu \, d\nu \quad (\text{erg cm}^{-2}\,\text{s}^{-1})$

$\Rightarrow \quad I = \int I_\nu \, d\nu \quad (\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{sr}^{-1})$

Energy Density and Specific Intensity

A very handy concept in astrophysics is to quantify the energy density associated with the radiation field. It can be shown that the energy density

$$u_\nu(\Omega) = \frac{I_\nu}{c} \qquad [\text{erg cm}^{-3}\,\text{sr}^{-1}\,\text{Hz}^{-1}]$$

One can say that the brightness or specific intensity

$$I_\nu = \underbrace{u_\nu(\Omega)\,c}_{} \qquad [\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{sr}^{-1}\,\text{Hz}^{-1}]$$

↳ Intensity is energy density propagating with $v = c$.

As before

$$u(\Omega) = \int u_\nu(\Omega)\, d\nu \qquad (\text{erg cm}^{-3}\,\text{sr}^{-1})$$

The total energy density of the radiation field is then

$$u_{Tot} = \int u(\Omega)\, d\Omega \qquad (\text{erg cm}^{-3})$$

The energy density can also be expressed in terms of a mean spatial intensity J_ν

$$\Rightarrow \quad u_\nu(\Omega) = \frac{I_\nu}{c}$$

$$u_\nu = \frac{1}{c}\int I_\nu\, d\Omega$$

$$= \frac{4\pi}{c} \times \frac{1}{4\pi}\int I_\nu\, d\Omega$$

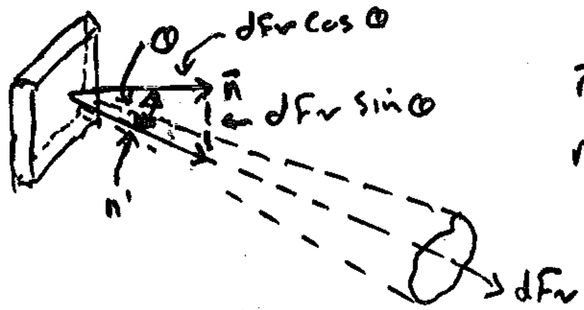
$$= \frac{4\pi}{c}\, J_\nu$$

$$J_\nu = \frac{1}{4\pi}\int I_\nu\, d\Omega$$

Roughly: $\quad J_\nu \approx I_\nu \dfrac{\Delta\Omega}{4\pi}$

Directionality of Radiation

Flux is always quantified as the radiation that cuts a surface perpendicularly. If a radiating slab is viewed from an angle θ the flux cutting the slab perpendicularly is:



$\bar{n} \perp dA$

$n' \to$ observer

$$dF_\nu(n) = dF_\nu \underbrace{\cos\theta}_{(\bar{n}\cdot\bar{n}')} \qquad [\text{Flux parallel to } n]$$

$$= \frac{dF}{d\nu}\cos\theta$$

$$= \frac{dE}{dA\,dt\,d\nu}\cos\theta$$

$$= F_{\perp,\nu}\cos\theta \qquad\qquad F_{\perp,\nu} = I_\nu\, d\Omega$$

$$= I_\nu\, d\Omega\cos\theta$$

$$= I_\nu\cos\theta\, d\Omega$$

$$F_\nu(n) = \int I_\nu\cos\theta\, d\Omega$$

Radiation Pressure

Pressure as Momentum Flux :

$$P_\perp = \frac{F}{A}$$

$$dP_\perp = \frac{(dp/dt)_\perp}{dA}$$

momentum transfer of photon

$$= \frac{dp}{dA\,dt}\cos\theta$$

A photon is a zero rest mass particle

$$E = \sqrt{m_0^2 c^4 + p^2 c^2}$$

$$= \sqrt{p^2 c^2}$$

$$= pc$$

$$p = \frac{E}{c} \Rightarrow dp = \frac{dE}{c}$$

$$dP_\perp = \frac{dp}{dA\,dt}\cos\theta = \frac{1}{c}\,\frac{dE}{dA\,dt}\cos\theta$$

$$dP_\perp = \frac{1}{c}\,dF\cos\theta$$

$$= \frac{1}{c}\left[I_\nu \cos\theta\, d\Omega\, d\nu\right]\cos\theta$$

$$= \frac{1}{c} I_\nu \cos^2\theta\, d\Omega\, d\nu$$

where

$$dF = dF(\theta)$$

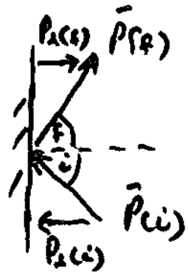
$$= I_\nu \cos\theta\, d\Omega\, d\nu$$

Therefore

$$dP_\perp = \frac{1}{c} I_\nu \cos^2\theta \, d\Omega \, d\nu$$

$$dP_{\nu\perp} = \frac{dP_\perp}{d\nu} = \frac{1}{c} I_\nu \cos^2\theta \, d\Omega$$

However one must add a correction factor here since for elastic collisions the momentum transfer in a single elastic collision



$[\hat{i} = \hat{f}]$

$$\Delta P_\perp = 2 P_\perp = 2 \frac{E_\perp}{c} = 2 \frac{E}{c} \cos\theta$$

$$\Delta P_\perp = \vec{P}_\perp(f) - \vec{P}_\perp(i) = P_\perp(f) - (-P_\perp(i)) = 2 P_\perp(f) = 2 \frac{E_\perp}{c} = 2 \frac{E}{c} \cos\theta$$

Therefore

$$dP_{\nu\perp} = \frac{2}{c} I_\nu \cos^2\theta \, d\Omega$$

$$dP_\perp = \frac{2}{c} I_\nu \cos^2\theta \, d\nu \, d\Omega$$

Therefore the total radiation pressure

$$\Rightarrow \quad P_\nu = \int dP_{\nu\perp} = \frac{2}{c} \int I_\nu \cos^2\theta \, d\Omega$$

$$\Rightarrow \quad P = \int P_\nu \, d\nu = \frac{2}{c} \int\int I_\nu \cos^2\theta \, d\Omega \, d\nu = \frac{2}{c} \int I_\nu \, d\nu \int \cos^2\theta \, d\Omega$$

This becomes

$$P = \frac{2}{c} \int I_\nu \, d\nu \int \cos^2\theta \, d\Omega$$

$$= 2 \times \left(\frac{1}{c} \int I_\nu \, d\nu\right) \int \cos^2\theta \, d\Omega$$

$$= 2 \times \int \frac{I_\nu}{c} \, d\nu \int \cos^2\theta \, d\Omega$$

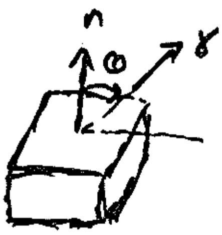
$$= \frac{2}{4\pi} \times \int 4\pi \frac{I_\nu}{c} \, d\nu \int \cos^2\theta \, d\Omega$$

$$= \frac{2}{4\pi} \times \int 4\pi \times U_\nu(\Omega) \, d\nu \int \cos^2\theta \, d\Omega$$

But :

$$U_\nu(\Omega) = \frac{U_\nu}{4\pi}$$

$$\boxed{\begin{aligned} P &= \frac{1}{2\pi} \times \underbrace{\int U_\nu \, d\nu}_{U_{Tot}} \int \cos^2\theta \, d\Omega \\ &= \frac{1}{2\pi} \times \int \cos^2\theta \, d\Omega \quad U_{Tot} \\ &= \frac{1}{2\pi} \times 2\pi/3 \; U_{Tot} \\ &= \frac{U_{Tot}}{3} \end{aligned}}$$



photons can escape for $(0 \leq \theta \leq \pi/2)$

Using: $d\Omega = \sin\theta \, d\theta \, d\phi$

$$\Rightarrow \int \cos^2\theta \, d\Omega = \int_0^{2\pi} \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \times \left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

$$= -\frac{2\pi}{3}\left[0-1\right] = \frac{2\pi}{3}$$

Radiation Transfer

When one observe a background source through a foreground plasma which also produce and absorb radiation, the net observed radiation is:

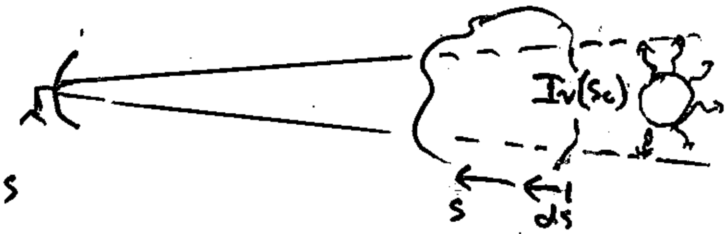
Emission: $dI_\nu^+ = j_\nu \, ds$ ← Emission coefficient of foreground plasma

Absorption: $dI_\nu^- = -\alpha_\nu I_\nu \, ds$ ← absorption coefficient of foreground plasma

$$\frac{dI_\nu}{ds} = \frac{dI_\nu^+}{ds} + \frac{dI^-}{ds}$$

$$= j_\nu - \alpha_\nu I_\nu$$

Emission only: $\alpha_\nu = 0$



$$dI_\nu = j_\nu \, ds$$

$$\int_{s_0}^{s} dI_\nu = \int_{s_0}^{s} j_\nu \, ds$$

$$I_\nu(s) - I_\nu(s_0) = \int_{s_0}^{s} j_\nu \, ds$$

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu \, ds$$

↑ Brightness of background source before the radiation enters foreground plasma

Absorption only : $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

$$\frac{dI_\nu}{I_\nu} = -\alpha_\nu ds$$

$$\int_{s_0}^{s} \frac{dI_\nu}{I_\nu} = -\int_{s_0}^{s} \alpha_\nu ds$$

$$\left[\ln I_\nu\right]_{s_0}^{s} = -\int_{s_0}^{s} \alpha_\nu ds$$

$$I_\nu(s) = I_\nu(s_0)\, e^{-\int \alpha_\nu ds}$$

For an absorbing medium the intensity decreases exponentially. The measure of absorption is determined by the absorption coefficient α_ν

This can conveniently be expressed in terms of the optical depth

$$\Rightarrow \tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu ds$$

$$\Rightarrow d\tau_\nu(s') = \alpha_\nu ds$$

$\tau_\nu > 1$: Source is optically thick (opaque)
$\tau_\nu < 1$: Source is optically thin (Transparent)

The radiation transfer eqn is then

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

$$\frac{dI_\nu}{\alpha_\nu ds} = \frac{j_\nu}{\alpha_\nu} - I_\nu$$

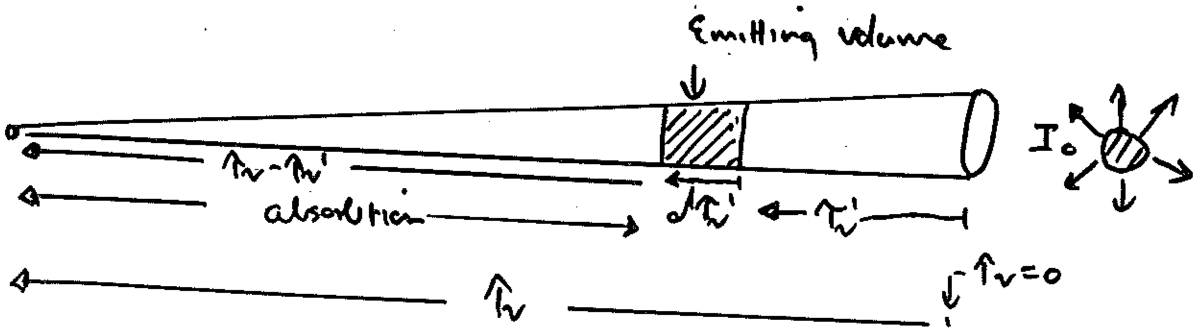
$$\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu$$

$$= S_\nu - I_\nu \qquad S_\nu = \frac{j_\nu}{\alpha_\nu}$$

$$= j_\nu l_\nu$$

↑ mean free path

An exact solution for this differential equation based on the scenario presented below:



Final solution:

- $$I_\nu(\tau_\nu) = I_\nu(\tau_\nu = 0)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau_\nu')\, e^{-(\tau_\nu - \tau_\nu')}\, d\tau_\nu'$$

For constant source function:

- $$I_\nu(\tau_\nu) = I_\nu(\tau_\nu = 0)\, e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-\tau_\nu}\, d\tau_\nu$$

$$= I_\nu(\tau_\nu = 0)\, e^{-\tau_\nu} + S_\nu \left[1 - e^{-\tau_\nu} \right]$$

Emission plus Absorption

Consider the radiation transfer eqn

$$I_\nu(\tau_\nu) = I_\nu(\tau_\nu=0)e^{-\tau_\nu} + S_\nu(1-e^{-\tau_\nu})$$

- Switching off background source $I_\nu(\tau_\nu=0) \to 0$

$$I_\nu(\tau_\nu) = S_\nu(1-e^{-\tau_\nu})$$
$$= \frac{j_\nu}{\alpha_\nu}(1-e^{-\tau_\nu})$$
$$= \frac{j_\nu R_s}{\alpha_\nu R_s}(1-e^{-\tau_\nu})$$
$$= j_\nu R_s\left(\frac{1-e^{-\tau_\nu}}{\tau_\nu}\right) \qquad \tau_\nu = \alpha_\nu R_s$$

For optically thin source $\tau_\nu \ll 1$

$$1-e^{-\tau_\nu} = 1-(1-\tau_\nu + \text{HOT})$$
$$= \tau_\nu$$

$$I_\nu(\tau_\nu) = j_\nu R_s\left(\frac{1-e^{-\tau_\nu}}{\tau_\nu}\right)$$
$$= \underbrace{j_\nu R_s}_{} \qquad \tau_\nu < 1$$

↳ The measured flux consist of the contribution from the whole foreground plasma

For optically thick source $\tau_\nu \gg 1$

$$I_\nu(\tau_\nu) = j_\nu R_s \left(\frac{1-e^{-\tau_\nu}}{\tau_\nu}\right)$$

$$\approx j_\nu \frac{R_s}{\tau_\nu} \qquad (\tau_\nu > 1)$$

$$\approx j_\nu \Delta R \qquad \Delta R \approx \frac{R_s}{\tau_\nu}$$

The only radiation that escapes the source comes from the outer shell of thickness

$$\Delta R \approx \frac{R_s}{\tau_\nu} \qquad \tau_\nu > 1$$

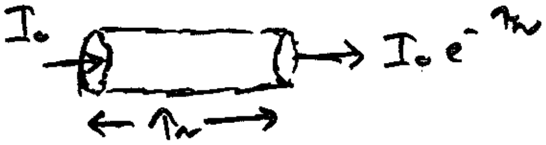
which is optically thin to the radiation produced inside it.

To summarize

- $\tau_\nu \ll 1$ $\qquad I_\nu(\tau_\nu) = j_\nu R_s$
- $\tau_\nu \gg 1$ $\qquad I_\nu(\tau_\nu) = j_\nu \dfrac{R_s}{\tau_\nu}$

The mean free path

The mean free path can be deduced from the absorption law of photons



The probability of a photon surviving the region with optical depth τ_ν

$$P_\nu = \frac{I_F}{I_0} = \frac{I_0 e^{-\tau_\nu}}{I_0} = e^{-\tau_\nu}$$

$\tau_\nu \to 0 \qquad P \to 1$

$\tau_\nu > 0 \qquad P < 1 \quad (P \to 0$ exponentially$)$

The average optical depth

$$\langle \tau_\nu \rangle = \frac{\int \tau_\nu P_\nu \, d\tau_\nu}{\int P_\nu \, d\tau_\nu}$$

$$= \frac{\int \tau_\nu e^{-\tau_\nu} \, d\tau_\nu}{\int e^{-\tau_\nu} \, d\tau_\nu}$$

But

$$\int_0^\infty e^{-\tau_\nu} \, d\tau_\nu = \left[-e^{-\tau_\nu}\right]_0^\infty = \left[-\frac{1}{\infty} + 1\right] = 1$$

The mean optical depth that will result in a certain interaction with matter

$$\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu e^{-\tau_\nu} \, d\tau_\nu = 1$$

Exercise: Show that $\langle \tau_\nu \rangle = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$

This results in

$$\langle \tau_\nu \rangle = \alpha_\nu l_\nu = 1$$

$$\Rightarrow \quad l_\nu = \frac{1}{\alpha_\nu} \qquad \text{mean free path}$$

$$= \frac{1}{n_T \sigma_\nu}$$

n_T: Target density; σ_ν: cross section

$$\Rightarrow l_\nu = \frac{R_s}{n_T \sigma_\nu R_s} = \frac{R_s}{\tau_\nu}$$

1.) $\tau_\nu > 1$ $\quad l_\nu < R_s$ (Radiation trapped)

2.) $\tau_\nu < 1$ $\quad l_\nu > R_s$ (Radiation can escape)

Random Walk

Consider a photon emitted in a homogeneous scattering region. Each random scatter is associated with a displacement vector $\bar{r}_i$. The net displacement

$$\bar{R} = \bar{r}_1 + \bar{r}_2 + \dots + \bar{r}_N$$

A rough estimate of the total displacement

$$l_*^2 = \langle R^2 \rangle$$

$$= \langle (\bar{r}_1 + \bar{r}_2 + \dots \bar{r}_N) \cdot (\bar{r}_1 + \bar{r}_2 + \dots + \bar{r}_N) \rangle$$

$$= \langle \bar{r}_1 \cdot \bar{r}_1 \rangle + \langle \bar{r}_2 \cdot \bar{r}_2 \rangle + \dots + \langle \bar{r}_N \cdot \bar{r}_N \rangle +$$

$$2 \langle \bar{r}_1 \cdot \bar{r}_2 \rangle + 2 \langle \bar{r}_1 \cdot \bar{r}_3 \rangle + \dots + 2 \langle \bar{r}_1 \cdot \bar{r}_N \rangle + \dots + 2 \langle \bar{r}_N \cdot \bar{r}_{N-1} \rangle$$

$$\approx \langle r_1^2 \rangle + \langle r_2^2 \rangle + \dots + \langle r_N^2 \rangle$$ since

All $\sum_{i \neq j} \langle \bar{r}_i \cdot \bar{r}_j \rangle = 0$ for $N \to \infty$

$$\Rightarrow l_*^2 = l_1^2 + l_2^2 + l_3^2 + \dots + l_N^2$$

$$\approx N l^2 \quad \text{(steps of equal size)}$$

$$l_* \approx \sqrt{N}\, l$$

For dense medium $(\tau >> 1)$

$$N = \frac{l_x^2}{l^2}$$

$$= \frac{L^2}{l^2}$$

$$= \left(\frac{L}{l}\right)^2$$

$$\approx \tau^2 \qquad [\text{for } \tau >> 1]$$

Let's motivate

$$l = \frac{1}{n_T \sigma}$$

$$N = \left(\frac{L}{l}\right)^2$$

$$= \left[\frac{L}{(1/n_T \sigma)}\right]^2$$

$$= (n_T \sigma \cdot L)^2$$

$$= \tau^2 \qquad (\tau = n_T \sigma L)$$

For dilute medium $\tau < 1$:

We showed earlier that the probability of a photon surveying optical depth τ_ν is

$$P(\tau_\nu) = e^{-\tau}$$

$$= (1 - \tau + \text{H.O.T})$$

The probability for an interaction is

$$P(\text{int}) = 1 - P(\tau)$$

$$= 1 - (1 - \tau + \text{HOT}) \qquad (\tau \ll 1)$$

$$\simeq \tau$$

$$= n_T \sigma L$$

$$= \left(\frac{L}{\ell}\right) \qquad \ell = \frac{1}{n_T \sigma}$$

For $\tau < 1$:

$$L \sim N \ell$$

$$\Rightarrow \quad N = \left(\frac{L}{\ell}\right) \approx \tau$$

Hence

$\tau \gg 1$: $N \approx \tau^2$

$\tau \ll 1$: $N \approx \tau$

$$N = \underset{\substack{\uparrow \\ (\tau \gg 1)}}{\tau^2} + \underset{\substack{\uparrow \\ (\tau \ll 1)}}{\tau}$$

Radiation force

We showed earlier

$$dP_\perp = \frac{1}{c}\,\frac{dE}{dA\,dt}\,\cos\theta$$

$$F_\perp = dP_\perp\, dA = \frac{1}{c}\,\frac{dE}{dt}\,\cos\theta$$ [$\perp$ component of force]

$$f_\perp = \frac{F_\perp}{dV} = \frac{1}{c}\,\frac{dE}{dV\,dt}\,\cos\theta$$

$$f_\perp = \frac{1}{c}\,\frac{dE}{dA\,\ell\,dt}\,\cos\theta$$ [ℓ = mean free path]

$$= \frac{1}{c} \times \frac{1}{\ell} \times \left(\frac{dE}{dA\,dt}\right) \cos\theta$$

$$= \frac{1}{c} \times \frac{1}{\ell} \times \underset{\uparrow\ \text{flux}}{F(\bar{n})}\ \cos\theta$$

$$\Rightarrow \quad f_\perp = f(\bar{n})\cos\theta$$

$$f(\bar{n}) = \frac{1}{c} \times \frac{1}{\ell} \times F(\bar{n})$$

$$= \frac{1}{c}\,\underset{\text{absorption coefficient}}{\alpha}\,F(\bar{n})$$

If there is spectral information:

$$f_\nu(\bar{n}) = \frac{1}{c} \times \alpha_\nu F_\nu(\bar{n}) \qquad \left[\alpha_\nu = \frac{1}{\ell_\nu}\right]$$

$$f(\bar{n}) = \frac{1}{c}\int \alpha_\nu F_\nu(\bar{n})\,d\nu \qquad \text{(force density)}$$

Force per unit mass:

$$\mathcal{F}_m = \frac{f(\bar{n})}{\rho} = \frac{1}{c}\int \kappa_\nu F_\nu(\bar{n})\,d\nu$$

Planck spectrum

For a $\tau_\nu \gg 1$ BB emitter

$$I_\nu = B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

Exercise: Show that

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}$$

The brightness of a BB emitter increases with temperature T (see p. 22 R & L, Fig 1.11).

- The higher the temperature the brighter the source
- One can also see from Fig 1.11 (p. 22) that for higher temperatures the maximum shifts to higher frequencies. The frequency where the emission peaks (for given T)

$$h\nu_{max} = 2.82 \; kT$$

$$\Rightarrow \quad \nu_{max} = 5.88 \times 10^{10} \, (Hz \; K^{-1}) \times T(K)$$

For example: For surface of Sun $T \sim 5800$ K

$$\nu_{max} \approx 3 \times 10^{14} \; Hz \left(\frac{T}{5800\,K}\right)$$

Asymptotic limits:

1.) $h\nu \ll kT$ $\qquad B_\nu(T) \approx \frac{2\nu^2}{c^2} kT$ (R-J)

2.) $h\nu \gg kT$ $\qquad B_\nu(T) = \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}$ (Wien)

For a BB emitter:

1.) $$F = \pi B$$
$$= \sigma T^4 \qquad \sigma = \frac{ac}{4} = 5.67\times10^{-5}\ \mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$$

2.) $$u = aT^4 \qquad a = \frac{4\sigma}{c} = 7.56\times10^{-15}\ \mathrm{erg\,cm^{-3}\,K^{-4}}$$

Brightness Temperature

The intensity of any source at a specific frequency can be compared to a BB with the same brightness at that frequency. The temperature producing the same brightness as our source is called the brightness temp T_b.

$$I_\nu = B_\nu(T_b)$$
$$= \frac{2\nu^2}{c^2} kT_b$$

Radiation Transfer and Temperature

We showed earlier that

$$I_\nu(\tau_\nu) = I_\nu(\tau_\nu=0)\, e^{-\tau_\nu} + S_\nu\left(1-e^{-\tau_\nu}\right)$$

↑ Source function of foreground plasma

If the background source is compared to a BB and foreground plasma's brightness is gauged with the BB flux

$$I_\nu(T_b) = \frac{2\nu^2}{c^2} k T_b \qquad (h\nu \ll kT)$$

$$I_\nu(T) = \frac{2\nu^2}{c^2} k T = S_\nu(T)$$

$$\Rightarrow \quad T_b(\tau_\nu) = T_b(\tau_\nu=0)\, e^{-\tau_\nu} + T\left(1-e^{-\tau_\nu}\right)$$

One can see that for $\tau_\nu \to \infty$

$$T_b(\tau_\nu) \to T$$

If we switch off background source and consider the case where $(0<\tau_\nu<1)$:

$$T_b(\tau_\nu) = T\left(1-e^{-\tau_\nu}\right)$$

$$T = \frac{T_b(\tau_\nu)}{1-e^{-\tau_\nu}} = \frac{T_b(\tau_\nu)}{[1-(1-\tau_\nu + \text{HOT})]} = \frac{T_b(\tau_\nu)}{\tau_\nu} \qquad [0<\tau_\nu<1]$$

- Optically thin cloud should be hotter to produce same flux as BB source at T_b.