Asymmetric filter convergence and completeness

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Abstract

Completeness for metric spaces is traditionally presented in terms of convergence of Cauchy sequences, and for uniform spaces in terms of Cauchy filters. Somewhat more abstractly, a uniform space is complete if and only if it is closed in every uniform space in which it is embedded, and so isomorphic to any space in which it is densely embedded. This is the approach to completeness used in the point-free setting, that is, for uniform and nearness frames: a nearness frame is said to be complete if every strict surjection onto it is an isomorphism.

Quasi-uniformities and quasi-nearnesses on biframes provide appropriate structures with which to investigate uniform and nearness ideas in the asymmetric context. In [9] a notion of completeness (called "quasi-completeness") was presented for quasi-nearness biframes in terms of suitable strict surjections being isomorphisms, and a quasi-completion was constructed for any quasi-nearness biframe.

In this paper we show that quasi-completeness can indeed be viewed in terms of the convergence of certain filters, namely, the regular Cauchy bifilters. We use the notion of a T-valued bifilter, which generalizes the characteristic function of a filter. An important tool is an appropriate composition for such bifilters. We show that the right adjoint of the quasi-completion is the universal regular Cauchy bifilter and use it to prove this characterization of quasi-completeness.

We also construct the so-called Cauchy filter quotient for a biframe using a quotient of the downset biframe that involves only the Cauchy, and not the regularity, condition. Like the quasi-completion, this provides a universal Cauchy bifilter; unlike the quasi-completion, this construction is functorial.

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1 Introduction

Completeness for metric spaces is traditionally presented in terms of convergence of Cauchy sequences, and for uniform spaces in terms of Cauchy filters. Somewhat more abstractly, a uniform space is complete if and only if it is closed in every uniform space in which it is embedded, and so isomorphic to any space in which it is densely embedded. This is the approach to completeness used in the point-free setting, that is, for uniform and nearness frames: a nearness frame is said to be complete if every strict surjection onto it is an isomorphism. (See [1], [2] and [6].)

An important aspect of our work in this paper is asymmetry. The fact that an asymmetric distance function has two natural underlying topologies leads to the study of bitopological spaces, and, in the point-free setting, to biframes. Quasi-uniformities and quasi-nearnesses on biframes provide appropriate structures with which to investigate uniform and nearness ideas in the asymmetric context.

In [9] a notion of completeness (called "quasi-completeness") was presented for quasi-nearness biframes in terms of suitable strict surjections being isomorphisms, and a quasi-completion was constructed for any quasi-nearness biframe.

A primary aim of this paper is to show that quasi-completeness can indeed be viewed in terms of the convergence of certain filters, namely, the regular Cauchy bifilters. Bifilters on biframes were first introduced in [13]; they were presented as filters on the total part, generated by their first and second parts. They can equally well be thought of as certain characteristic functions to the **2**-chain; replacing **2** by an arbitrary biframe T leads to the notion of a T-

valued bifilter and it is this more general concept of a bifilter that is needed here. An important tool is an appropriate composition for bifilters, which in the symmetric case is, of course, a trivial matter.

In [13] we showed that the right adjoint of the join map from the downset biframe is the "universal" bifilter; loosely speaking, this means that any other bifilter on the same domain factors via the universal bifilter. Here we show that the right adjoint of the quasi-completion is the universal regular Cauchy bifilter and use this to prove the characterization of quasi-completeness mentioned above.

It is also of interest to drop the regularity condition and consider Cauchy bifilters in their own right. We construct the so-called Cauchy filter quotient for a biframe using a quotient of the downset biframe that involves only the Cauchy, and not the regularity, condition. Like the quasi-completion, this provides a universal Cauchy bifilter; unlike the quasi-completion, this construction is functorial. In order to obtain functoriality of the quasi-completion for certain biframes, we require a notion of having enough regular Cauchy bifilters. This is discussed in a subsequent paper.

2 Background

See [19], [16], [14], [23] and [18] as references for frame theory; see [3], [4], [10], [11], [20], [21] and [22] for biframes and related ideas and see [7], [8], [9], [12], [15] and [17] for information on structured biframes.

Frames and biframes

1. A *frame* L is a complete lattice in which the distributive law

$$x \wedge \bigvee \{y : y \in Y\} = \bigvee \{x \wedge y : y \in Y\}$$

holds for all $x \in L, Y \subseteq L$. A *frame map* is a set function between frames which preserves finite meets and arbitrary joins, and thus also the top (denoted 1) and the bottom (denoted 0) of the frame.

2. A *biframe* L is a triple $L = (L_0, L_1, L_2)$ in which L_0 is a frame, L_1 and L_2 are subframes of L_0, and $L_1 \cup L_2$ generates L_0. We call L_0 the *total part*, L_1 the *first part* and L_2 the *second part* of the biframe L. A *biframe map* $h : M \to L$ is a frame map from M_0 to L_0 such that the image of M_i under h is contained in L_i for $i = 1, 2$. We call the restriction $h|M_0$ the *total part* of the map h and $h|M_1 = h_1$ and $h|M_2 = h_2$ its *first* and *second parts* respectively. The resulting category of biframes and biframe maps is denoted by **BiFrm**. (In what follows, we will reserve the subscript i for reference to first and second parts only.)

3. A biframe map $h : M \to L$ is *dense* if its total part is a dense frame map, i.e. $a = 0$ whenever $h(a) = 0$, for any $a \in M_0$. It is *onto* if its first and second parts are onto. (h is then onto on the total part.)

4. For a biframe map $h : M \to L$ we define the right adjoint of h as the right adjoint of its total part. There is no *a priori* reason that r should map elements of L_i to elements of M_i. If $r[L_i] \subseteq M_i$ for $i = 1, 2$ we say that r is *part-preserving*.

Paircovers and quasi-nearnesses

We now present the basic definitions required for paircovers and quasi-nearness biframes. See [8] for an introduction to quasi-uniform biframes and [9] for an introduction to quasi-nearness biframes.

Definition 2.1 Let $L = (L_0, L_1, L_2)$ be a biframe.

1. $C \subseteq L_1 \times L_2$ is a *paircover* of L if $\bigvee\{c \wedge \tilde{c} : (c, \tilde{c}) \in C\} = 1$.

2. A paircover C of L is *strong* if, for any $(c, \tilde{c}) \in C$, whenever $c \wedge \tilde{c} = 0$ then $c \vee \tilde{c} = 0$, that is, $(c, \tilde{c}) = (0, 0)$. For an arbitrary paircover C, the paircover $C^r = \{(c, \tilde{c}) \in C : c \wedge \tilde{c} \neq 0\}$ is strong.

3. For any paircovers, C and D of L, we write $C \leqslant D$ if for any $(c, \tilde{c}) \in C$ there is $(d, \tilde{d}) \in D$ with $c \leqslant d$ and $\tilde{c} \leqslant \tilde{d}$ (for which we write $(c, \tilde{c}) \leq (d, \tilde{d})$). We then say C *refines* D. We also set $C \wedge D = \{(c \wedge d, \tilde{c} \wedge \tilde{d}) : (c, \tilde{c}) \in C, (d, \tilde{d}) \in D\}$ which is also a paircover of L.

4. For $a \in L_0$ and C a paircover of L, we set
   $C_1 a = \bigvee\{c : (c, \tilde{c}) \in C \text{ and } \tilde{c} \wedge a \neq 0\}$ and
   $C_2 a = \bigvee\{\tilde{c} : (c, \tilde{c}) \in C \text{ and } c \wedge a \neq 0\}$.

Definition 2.2 Let $L = (L_0, L_1, L_2)$ be a biframe.

1. A non-empty family, $\mathcal{U}L$, of paircovers of L is a *quasi-nearness on* L if
   (a) The family of strong members of $\mathcal{U}L$ is a filter-base for $\mathcal{U}L$ with respect to $\wedge$ and $\leqslant$. (Filter condition)
   (b) For each $x \in L_i$, $x = \bigvee\{y \in L_i : \text{ for some } C \in \mathcal{U}L, C_i y \leqslant x\}$, $(i = 1, 2)$ (Compatibility condition)
2. For $x, y \in L_i$ we write $y \triangleleft_i x$ whenever there exists a $C \in \mathcal{U}L$ with $C_i y \leq x$. We write $(a, b) \triangleleft (c, d)$ when $a \triangleleft_1 c$ and $b \triangleleft_2 d$. We note that if $y \triangleleft_i x$ then $y \prec_i x$, that is, there is a $t \in L_k, k \neq i$ such that $y \wedge t = 0$ and $t \vee x = 1$.
3. The pair $(L, \mathcal{U}L)$ is called a *quasi-nearness biframe*. Members of $\mathcal{U}L$ will be referred to as *uniform* paircovers.
4. Let $(L, \mathcal{U}L)$ and $(M, \mathcal{U}M)$ be quasi-nearness biframes. A biframe map $f : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ is *uniform* if for every $C \in \mathcal{U}M$, $f[C] \in \mathcal{U}L$, where $f[C] = \{(f(c), f(\tilde{c})) : (c, \tilde{c}) \in C\}$.
5. Quasi-nearness biframes and uniform maps are the objects and arrows of the category **QNearBiFrm**.

There is an appropriate notion of star-refinement for paircovers. As may be expected, a quasi-nearness biframe in which every uniform paircover has a uniform star-refinement is called a quasi-uniform biframe. In this paper star-refinements will not be needed.

Quasi-completeness and quasi-completions

The following appear in [9]:

Definition 2.3 Let $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ be a uniform map between quasi-nearness biframes and let r be the right adjoint of the total part of h.

1. We call r *uniformly-generating* if $r[D] \in \mathcal{U}M$ for each $D \in \mathcal{U}L$ and $\{r[D] : D \in \mathcal{U}L\}$ generates $\mathcal{U}M$. We remind the reader that r need not be a biframe map; indeed it is usually not. Note that if r is uniformly-generating it is also part-preserving. (See Lemma 5.2 in [9].)

2. We say that h is a *strict surjection* if r is uniformly-generating. Note that if h is a strict surjection then it is also dense, onto and $\{h[C] : C \in \mathcal{U}M\}$ generates $\mathcal{U}L$. (See Lemmas 5.3 and 5.6 in [9].)

3. $(L,\mathcal{U}L)$ is *quasi-complete* if every strict surjection $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ is an isomorphism.

4. A *quasi-completion* of $(L,\mathcal{U}L)$ is a strict surjection $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ where $(M,\mathcal{U}M)$ is a quasi-complete quasi-nearness biframe.

In [9] it is further shown that, for any quasi-nearness biframe $(L,\mathcal{U}L)$ there exists a quasi-completion $(CL, C\mathcal{U}L)$ and quasi-completion map $\gamma : (CL, C\mathcal{U}L) \to (L,\mathcal{U}L)$.

The downset biframe

The construction of the quasi-completion mentioned above depends heavily on the downset biframe. In this paper we will use the downset biframe again to construct the so-called "Cauchy quotient" for a biframe.

We assume that the reader is familiar with the frame of downsets of a frame L, usually denoted by $\mathbb{D}L$. (See for instance [19] for details.) It is well known that the join map $\bigvee : \mathbb{D}L \to L$ is a frame map and has right adjoint $\downarrow: L \to \mathbb{D}L$ given by $\downarrow x = \{y \in L : y \leq x\}$, for all $x \in L$.

Definition 2.4 ([9])

1. For a biframe L and $i = 1, 2$, let

$$(\mathbb{D}L)_i = \{U \in \mathbb{D}L_0 : U \text{ is generated by } U \cap L_i\},$$

   where we say that U is *generated by* $U \cap L_i$ if, for all $x \in U$, there is $y \in U \cap L_i$ such that $x \leq y$.

2. Then let $(\mathbb{D}L)_0$ be the subframe of $\mathbb{D}(L_0)$ generated by $(\mathbb{D}L)_1 \cup (\mathbb{D}L)_2$. Explicitly, any $U \in (\mathbb{D}L)_0$ can be written as

$$U = \bigcup\{\downarrow x \cap \downarrow y : x \in L_1, y \in L_2, x \wedge y \in U\}.$$

   This construction makes $\mathbb{D}L = ((\mathbb{D}L)_0, (\mathbb{D}L)_1, (\mathbb{D}L)_2)$ a biframe, called the *downset biframe of* L.

3. For a biframe map $h : L \to M$ and $U \in (\mathbb{D}L)_0$, define

$$\mathbb{D}h(U) = \bigcup\{\downarrow h(x) \cap \downarrow h(y) : x \in L_1, y \in L_2, x \wedge y \in U\}$$

   Then $\mathbb{D}h : \mathbb{D}L \to \mathbb{D}M$ is a biframe map.

4. The join map $\bigvee : \mathbb{D}L \to L$ is a biframe map which is dense and onto.

5. Let s be the right adjoint of $\bigvee : \mathbb{D}L \to L$. Then $s(a) = \bigcup\{U \in (\mathbb{D}L)_0 : U \subseteq \downarrow a\}$, for any $a \in L_0$. Note that if $a \in L_i$ for $i = 1, 2$ then $s(a) = \downarrow a$ since then $\downarrow a \in (\mathbb{D}L)_0$. This makes s part-preserving.

A factorization lemma

The following straightforward result will be useful in what follows.

Definition 2.5 For any biframe map $f : L \to M$, the *kernel* of f, written $\ker(f)$, is defined as $\{(a, b) \in L_0 \times L_0 : f(a) = f(b)\}$.

Lemma 2.6 Suppose that $f : L \to M$ is an onto biframe map and $g : L \to N$ a biframe map such that $\ker(f) \subseteq \ker(g)$. Then there is a unique biframe map $h : M \to N$ such that $hf = g$, that is, so that the following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, "\exists! h"] \\ N & \end{tikzcd}$$

PROOF. Consider the two biframe maps f and g as frame maps on total parts. Since f is onto and $\ker(f) \subseteq \ker(g)$, by the well-known result for frame maps, there is a unique frame map $h : M_0 \to N_0$ such that $hf = g$. We check that h is in fact a biframe map: Take $x \in M_i$ for $i = 1, 2$. Since f is onto as a biframe map, there is $y \in L_i$ with $f(y) = x$. Then $h(x) = hf(y) = g(y) \in N_i$, since g is a biframe map. ∎

3 Congruences describing nuclei on the downset biframe

In [9] a quasi-completion was constructed using a nucleus on the downset biframe that satisfies conditions (N1), (N2) and (N3) of Lemma 3.2 below. It will be important in what follows to consider the associated congruence, which we describe in this section.

Lemma 3.1 Let L be a biframe and n any nucleus on $(\mathbb{D}L)_0$, the total part of its downset biframe. The following conditions are equivalent:

1. $n(U) \subseteq \downarrow(\bigvee U)$ for any $U \in (\mathbb{D}L)_0$.
2. $\bigvee n(U) = \bigvee U$ for any $U \in (\mathbb{D}L)_0$.

Moreover, if n satisfies the first condition, then $n(\downarrow a) = \downarrow a$ for any $\downarrow a \in (\mathbb{D}L)_0$, and so, in particular, for any $a \in L_1 \cup L_2$.

PROOF. Suppose that $U \in (\mathbb{D}L)_0$.
(1) $\Rightarrow$ (2): $\bigvee n(U) \leq \bigvee \downarrow(\bigvee U) = \bigvee U$.
(2) $\Rightarrow$ (1): $n(U) \subseteq \downarrow \bigvee n(U) = \downarrow(\bigvee U)$.
Now suppose that n satisfies condition 1. For $x \in L_1 \cup L_2$, $\downarrow x \in (\mathbb{D}L)_0$, so $n(\downarrow x) \subseteq \downarrow(\bigvee \downarrow x) = \downarrow x$ and so $n(\downarrow x) = \downarrow x$. Moreover, if $a \in L_0$ and $\downarrow a \in (\mathbb{D}L)_0$ then $a = x \wedge y$ for some $x \in L_1, y \in L_2$ (see Lemma 3.11 in [13]). Then $n(\downarrow a) = n(\downarrow(x \wedge y)) = n(\downarrow x \cap \downarrow y) = n(\downarrow x) \cap n(\downarrow y) = \downarrow x \cap \downarrow y = \downarrow a$. ∎

We note for the record that the downset $\downarrow(\bigvee U)$ of L_0 above is not in general a member of $(\mathbb{D}L)_0$; in fact, the proof above indicates which principal downsets are in $(\mathbb{D}L)_0$.

Lemma 3.2 Let $(L, \mathcal{U}L)$ be a quasi-nearness biframe and let n be the smallest nucleus on $(\mathbb{D}L)_0$ satisfying the conditions:

(N1) $n(U) \subseteq \downarrow(\bigvee U)$ for all $U \in (\mathbb{D}L)_0$.

(N2) $n(\bigcup \widehat{C}^s) = \downarrow 1$ for all $C \in \mathcal{U}L$, where $\widehat{C}^s = \{\downarrow c \cap \downarrow \tilde{c} : (c, \tilde{c}) \in C\}$.

(N3) $n(k_i(a)) = \downarrow a$ for all $a \in L_i, i = 1, 2$, where
$k_i(a) = \{x \in L_0 : x \leq y \lhd_i a \text{ for some } y \in L_i\}$.

The congruence on $(\mathbb{D}L)_0$ corresponding to the nucleus n is generated by the pairs:
$\{(\bigcup \widehat{C}^s, \downarrow 1) : C \in \mathcal{U}L\} \cup \{(k_1(a), \downarrow a) : a \in L_1\} \cup \{(k_2(b), \downarrow b) : b \in L_2\}$.

PROOF. Let θ_n denote the congruence corresponding to the nucleus n and let γ be the congruence generated by the pairs $\{(\bigcup \widehat{C}^s, \downarrow 1) : C \in \mathcal{U}L\} \cup \{(k_1(a), \downarrow a) : a \in L_1\} \cup \{(k_2(b), \downarrow b) : b \in L_2\}$. Since n satisfies (N2) and (N3), θ_n contains the generating pairs of γ and so $\gamma \subseteq \theta_n$.
To show that $\theta_n \subseteq \gamma$, we show that $n \leq n_\gamma$, where n_γ denotes the nucleus corresponding to γ. It suffices to show that n_γ satisfies (N1) to (N3), since n is the smallest nucleus satisfying these conditions.
(N1): By Lemma 3.1, it suffices to show that $\bigvee n_\gamma(U) = \bigvee U$ for any $U \in (\mathbb{D}L)_0$. Let β be the kernel of the join map $\bigvee : (\mathbb{D}L)_0 \to L_0$. In fact β

contains the generating pairs of γ, since the join map identifies all such, so we have $\gamma \subseteq \beta$. So $n_\gamma \leq n_\beta$. For $U \in (\mathbb{D}L)_0$, $\bigvee n_\gamma(U) \leq \bigvee n_\beta(U) = \bigvee U$, the latter equality holding since $n_\beta(U)$ is the largest member of $(\mathbb{D}L)_0$ having the same join as U.
(N2): For $C \in \mathcal{U}L$, $n_\gamma(\bigcup \widehat{C}^s) = \downarrow 1$.
(N3): For $a \in L_i, i = 1, 2$, $n_\gamma(k_i(a)) = n_\gamma(\downarrow a) = \downarrow a$. ∎

The condition (N2) can be thought of as a "Cauchy" condition, and (N3) as a "regularity" condition. We will also, in this paper, have need for the quotient of $(\mathbb{D}L)_0$ that imposes just the Cauchy condition and (N1); the relevant nucleus and congruence are given in Lemma 3.3 below.

Lemma 3.3 Let $(L, \mathcal{U}L)$ be a quasi-nearness biframe and let m be the smallest nucleus on $(\mathbb{D}L)_0$ satisfying the conditions (N1) and (N2) of Lemma 3.2. The congruence on $(\mathbb{D}L)_0$ corresponding to the nucleus m is generated by the pairs $\{(\bigcup \widehat{C}^s, \downarrow 1) : C \in \mathcal{U}L\}$.

PROOF. Apply the proof of Lemma 3.2, omitting all reference to the condition (N3). ∎

4 Composition of bifilters

In [13] we introduced the notion of a bifilter on a biframe: it is a subset of the given biframe with certain closure properties. This was generalized to the notion of a T-valued bifilter on a biframe. It is this general form of a bifilter that we use almost exclusively in this paper, and we recall here the main facts about such bifilters. We will omit the words "general" and "T-valued" after giving the definition.

Definition 4.1 1. For biframes L and T, a function $\varphi : L_0 \to T_0$ is a *general T-valued bifilter on L* if:

- φ preserves $0, \wedge$ and 1.
- $\varphi[L_i] \subseteq T_i$ for $i = 1, 2$, that is, φ is part-preserving.

- For any $a \in L_0$, $\varphi(a) = \bigvee\{\varphi(x \wedge y) : x \in L_1, y \in L_2, x \wedge y \leq a\}$.

(We say for the last condition that " φ is determined by its action on first and second parts". We also speak loosely of a "bifilter $\varphi : L \to T$" without causing any confusion.)

2. For a quasi-nearness biframe $(L, \mathcal{U}L)$ a bifilter $\varphi : L \to T$ is *Cauchy* if, for any $C \in \mathcal{U}L$, $\varphi[C]$ is a paircover of T, that is, $\bigvee\{\varphi(c \wedge \tilde{c}) : (c, \tilde{c}) \in C\} = 1$.

3. For a quasi-nearness biframe $(L, \mathcal{U}L)$ a bifilter $\varphi : L \to T$ is *regular* if, for $x \in L_i, i = 1, 2$, $\varphi(x) = \bigvee\{\varphi(z) : z \in L_i, z \triangleleft_i x\}$.

The definitions above are motivated by the work of Banaschewski, Hong and Pultr in [5].

Note 4.2

1. We note that, since any bifilter is determined by its action on first and second parts, it is easy to see that if two bifilters φ and ψ agree on first and second parts, then $\varphi = \psi$. Similarly, if $\varphi(x) \leq \psi(x)$ for $x \in L_1 \cup L_2$, then $\varphi \leq \psi$.

2. Any biframe map between biframes is a bifilter.

3. Any biframe map (not necessarily uniform) between quasi-nearness biframes is a regular Cauchy bifilter.

In the setting of frames, the usual function composition of two general filters is again, trivially, a filter. The situation for bifilters is not so simple. We now define a composition of bifilters that does indeed produce a bifilter.

Definition 4.3 Suppose that $\varphi : L \to M$ and $\rho : M \to T$ are bifilters. We define their composite by

$$\rho \bullet \varphi\ (a) = \bigvee \left\{\rho\varphi(x) \wedge \rho\varphi(y) : x \in L_1, y \in L_2, x \wedge y \leq a\right\}.$$

We note that the usual function composition will always be denoted by juxtaposition. The special case where φ is a biframe map was treated in [13].

Lemma 4.4 Suppose that $\varphi : L \to M$ and $\rho : M \to T$ are bifilters. Then $\rho{\bullet}\varphi : L \to T$ is a bifilter and $\rho{\bullet}\varphi\,(x) = \rho\varphi(x)$ for any $x \in L_1 \cup L_2$.

PROOF. Note that, for $x \in L_1 \cup L_2$, $(\rho{\bullet}\varphi)(x) = \rho\varphi(x)$ since $\rho\varphi(x) = \rho\varphi(x) \wedge 1 = \rho\varphi(x) \wedge \rho\varphi(1)$.
It is clear that $\rho{\bullet}\varphi\,(0) = 0$ and $\rho{\bullet}\varphi\,(1) = 1$ and that $\rho{\bullet}\varphi$ is order-preserving.
For $a, b \in L_0$,

$$\begin{aligned}
& \rho{\bullet}\varphi\,(a) \wedge \rho{\bullet}\varphi\,(b) \\
= \ & \bigvee\{\rho\varphi(s) \wedge \rho\varphi(t) : s \in L_1, t \in L_2, s \wedge t \leq a\} \\
& \wedge \bigvee\{\rho\varphi(u) \wedge \rho\varphi(v) : u \in L_1, v \in L_2, u \wedge v \leq b\} \\
= \ & \bigvee\{\rho\varphi(s \wedge u) \wedge \rho\varphi(t \wedge v) : s, u \in L_1, t, v \in L_2, s \wedge t \leq a, u \wedge v \leq b\} \\
\leq \ & \rho{\bullet}\varphi\,(a \wedge b)
\end{aligned}$$

and since $\rho{\bullet}\varphi$ is order-preserving, $\rho{\bullet}\varphi\,(a \wedge b) = \rho{\bullet}\varphi\,(a) \wedge \rho{\bullet}\varphi\,(b)$.
Finally, for $a \in L_0$,

$$\rho{\bullet}\varphi\,(a) = \bigvee\{\rho{\bullet}\varphi\,(x) \wedge \rho{\bullet}\varphi\,(y) : x \in L_1, y \in L_2, x \wedge y \leq a\},$$

so $\rho{\bullet}\varphi$ is determined by its action on first and second parts. ∎

We now give some simple but frequently used results concerning composition of bifilters.

Lemma 4.5 Suppose that $\varphi : L \to M$ and $\rho : M \to T$ are bifilters, with $\rho{\bullet}\varphi$ as defined above.

1. If ρ is a biframe map, then $\rho{\bullet}\varphi = \rho\varphi$, since ρ preserves joins.
2. If φ is a uniform map between quasi-nearness biframes and ρ is a Cauchy bifilter, then $\rho{\bullet}\varphi$ is Cauchy.
3. If φ is a Cauchy bifilter on a quasi-nearness biframe L and ρ is a biframe map, then $\rho{\bullet}\varphi$ is Cauchy.

4. If φ is a regular bifilter and ρ is a biframe map then $\rho_\bullet\varphi$ is regular.

PROOF. All the proofs are straightforward. ∎

5 Three universal bifilters

We now present, on any given (quasi-nearness) biframe L, three bifilters that are each universal in a sense to be explained below.

Definition 5.1 Let L be a biframe.
(1) An S-valued bifilter α on L is said to be the *universal bifilter on* L iff for any T-valued bifilter φ on L there is a unique biframe map $\bar{\varphi} : S \to T$ such that $\bar{\varphi}\alpha = \varphi$, that is, the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & S \arrow[dl, "\bar{\varphi}"] \\ T & \end{tikzcd}$$

(2) Similarly, if $(L,\mathcal{U}L)$ is a quasi-nearness biframe, an S-valued bifilter α on $(L,\mathcal{U}L)$ is the *universal Cauchy bifilter* (resp. *universal regular Cauchy bifilter*) *on* $(L,\mathcal{U}L)$ iff for any T-valued Cauchy (resp. regular Cauchy) bifilter φ on $(L,\mathcal{U}L)$ there is a unique biframe map $\bar{\varphi} : S \to T$ such that $\bar{\varphi}\alpha = \varphi$.

First universal bifilter: In [13], Proposition 6.8, it was shown that for any biframe L the right adjoint, s, of the join map $\bigvee : \mathbb{D}L \to L$ is the universal bifilter on L.

Second universal bifilter: In [9] a quasi-completion $\gamma : (CL, C\mathcal{U}L) \to (L,\mathcal{U}L)$ was constructed, for any quasi-nearness biframe $(L,\mathcal{U}L)$. Its right adjoint, denoted by σ, will be shown (see Proposition 5.5 in this paper) to be the universal regular Cauchy bifilter on $(L,\mathcal{U}L)$.

In order to provide the third universal bifilter, namely a universal Cauchy bifilter for a quasi-nearness biframe $(L,\mathcal{U}L)$, we need to construct CFL, which is the quotient of the downset biframe $\mathbb{D}L$ obtained by imposing a "Cauchy" condition as discussed before Lemma 3.3. Its construction is similar to that of the quasi-completion provided in [9] (see page 524 there) so we provide only a definition and a brief summary of its properties.

Let $(L,\mathcal{U}L)$ be a quasi-nearness biframe. Let m be the nucleus on $(\mathbb{D}L)_0$ defined in Lemma 3.3. We note that m is obtained by taking the meet of all nuclei on $(\mathbb{D}L)_0$ satisfying (N1) and (N2).

Now consider m as the canonical frame map to the quotient frame of fixed objects of $(\mathbb{D}L)_0$ under m, so $m : (\mathbb{D}L)_0 \to m((\mathbb{D}L)_0)$. This then extends to a biframe map $m : ((\mathbb{D}L)_0, (\mathbb{D}L)_1, (\mathbb{D}L)_2) \to (m(\mathbb{D}L)_0, m(\mathbb{D}L)_1, m(\mathbb{D}L)_2)$ which we denote by $m : \mathbb{D}L \to \operatorname{Fix} m$.

Definition 5.2 We define $CFL = \operatorname{Fix} m$ and write $(CFL)_i = m(\mathbb{D}L)_i$.

Consider the following diagram:

$$\begin{tikzcd}
\mathbb{D}L \arrow[r, "\bigvee"] \arrow[d, "m"'] & L \\
CFL \arrow[ur, "g"'] &
\end{tikzcd}$$

We note the following:

- g is defined as the restriction of the join map to members of CFL.
- g is a dense, onto biframe map.

- The diagram above commutes, that is, $gm = \bigvee$.
- We define δ to be the right adjoint of g.
- δ preserves (arbitrary) meets, 0 and 1.
- $\delta(a) = \downarrow a$ for $a \in L_1 \cup L_2$, so δ is part-preserving.
- $\delta[L_i]$ generates $(CFL)_i$ for $i = 1, 2$.
- δ is determined by its action on first and second parts, as the following shows:
  For $a \in L_0$:

$$\begin{aligned}
&\delta(a) = \bigvee \{U \cap V : U \in (CFL)_1, V \in (CFL)_2, U \cap V \subseteq \delta(a)\} \\
&= \bigvee \Big\{ \bigvee_\beta \delta(x_\beta) \wedge \bigvee_\gamma \delta(y_\gamma) : x_\beta \in L_1, y_\gamma \in L_2, \delta(x_\beta) \subseteq U, \delta(y_\gamma) \subseteq V, U \cap V \subseteq \delta(a)\Big\} \\
&= \bigvee \bigvee_{\beta,\gamma} \big\{\delta(x_\beta) \wedge \delta(y_\gamma) : x_\beta \in L_1, y_\gamma \in L_2, \delta(x_\beta) \subseteq U, \delta(y_\gamma) \subseteq V, U \cap V \subseteq \delta(a)\big\}
\end{aligned}$$

  This is of the correct form, since for such x_β and y_γ,

$$x_\beta \wedge y_\gamma = g\delta(x_\beta \wedge y_\gamma) \leq g\delta(a) = a.$$

- For $C \in \mathcal{U}L$, $\delta[C]$ is a paircover of CFL by virtue of (N2).
- From the properties above, it follows that δ is indeed a Cauchy bifilter on $(L, \mathcal{U}L)$.
- We call CFL the Cauchy filter quotient for L.

Third universal bifilter: We will see in Proposition 5.6 that $\delta : L \to CFL$ is the universal Cauchy bifilter on L.

We now turn to the proofs of the fact that σ and δ are indeed universal.

Lemma 5.3 Let $h : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ be a strict surjection between quasi-nearness biframes. Its right adjoint, r, is a regular Cauchy bifilter.

PROOF. We recall the relevant properties of r from [9]:

- Since r is a right adjoint, it preserves (arbitrary) meets and 1.
- Since h is dense, r preserves 0.
- r is part-preserving ([9], Lemma 5.2)
- For $x \in L_i, i = 1, 2, r(x) = \bigvee\{r(z) : z \in L_i, z \lhd_i x\}$. ([9], Lemma 5.9)
- For $C \in \mathcal{U}L, r[C] \in \mathcal{U}M$ and so $r[C]$ is certainly a paircover of M.
- For $a \in L_0, r(a) = \bigvee\{r(x) \wedge r(y) : x \in L_1, y \in L_2, x \wedge y \leq a\}$ by an argument similar to the one shown for δ earlier.

∎

Corollary 5.4 For any quasi-nearness biframe $(L,\mathcal{U}L)$, the right adjoint, σ, of the quasi-completion $\gamma : (CL, C\mathcal{U}L) \to (L,\mathcal{U}L)$ is a regular Cauchy bifilter.

PROOF. The previous lemma applies because γ is a strict surjection. ∎

Proposition 5.5 For any quasi-nearness biframe $(L,\mathcal{U}L)$, $\sigma : L \to CL$ is the universal regular Cauchy bifilter on $(L,\mathcal{U}L)$.

PROOF. By the previous corollary, σ is a regular Cauchy bifilter. It remains to show universality.

First we note that, from the construction of $\gamma : CL \to L$, we have that $\gamma n = \bigvee$, where $\bigvee : \mathbb{D}L \to L$ and n is the nucleus used in defining CL, that is, the nucleus n defined in Lemma 3.2. Letting t be the right adjoint of n, we obtain $t\sigma = s$ and so $\sigma = n(t\sigma) = ns$.

Let $\varphi : L \to T$ be a regular Cauchy bifilter on $(L,\mathcal{U}L)$. Since $s : L \to \mathbb{D}L$ is a universal bifilter ([13] Proposition 6.8), there exists a unique biframe map $\bar{\varphi} : \mathbb{D}L \to T$ such that $\bar{\varphi}s = \varphi$. Explicitly, $\bar{\varphi}(U) = \bigvee\{\varphi(x) : x \in U\}$, for all $U \in (\mathbb{D}L)_0$.

In the diagram below, we seek a biframe map $\widetilde{\varphi} : CL \to T$ such that $\widetilde{\varphi}\sigma = \varphi$:

$$
\begin{tikzcd}
L \arrow[rr, "\sigma"] \arrow[dr, "s"] \arrow[dd, "\varphi"'] & & CL \arrow[ddll, bend left, "\widetilde{\varphi}"] \\
& \mathbb{D}L \arrow[ur, "n"'] \arrow[dl, "\bar{\varphi}"'] & \\
T & &
\end{tikzcd}
$$

We use Lemma 2.6. To apply this, we use the fact that n is an onto biframe map, and must check that the kernel of n is contained in the kernel of $\bar{\varphi}$. To that end, we calculate the following:
For $a \in L_i$,

$$
\begin{aligned}
\bar{\varphi}(k_i(a)) &= \bigvee\{\varphi(x) : x \in k_i(a)\} \\
&= \bigvee\{\varphi(z) : z \in L_i, z \lhd_i a\} \\
&= \varphi(a) \\
&= \bar{\varphi}(s(a)) \\
&= \bar{\varphi}(\downarrow a)
\end{aligned}
$$

The third equality uses the regularity condition of φ, while the last equality uses the fact that $s(a) = \downarrow a$ for $a \in L_i, i = 1, 2$.

For $C \in \mathcal{U}L$,

$$
\begin{aligned}
\bar{\varphi}(\bigcup \widehat{C}^s) &= \bigvee\{\varphi(x) : x \in \bigcup \widehat{C}^s\} \\
&= \bigvee\{\varphi(c) \wedge \varphi(\tilde{c}) : (c, \tilde{c}) \in C\} \\
&= 1 \\
&= \bar{\varphi}(\downarrow 1)
\end{aligned}
$$

The third equality uses the Cauchy condition of φ.

So the kernel of $\bar{\varphi}$ contains the generating pairs of the kernel of n (see Lemma 3.2), and so $\ker(n) \subseteq \ker(\bar{\varphi})$.

By Lemma 2.6, there exists a biframe map $\widetilde{\varphi} : CL \to T$ such that $\widetilde{\varphi}n = \bar{\varphi}$. Then

$$\widetilde{\varphi}\sigma = \widetilde{\varphi}(ns) = \bar{\varphi}s = \varphi,$$

as required. We note that $\widetilde{\varphi}$ is the unique such map, because $\sigma[L]$ generates CL. ∎

Proposition 5.6 For any quasi-nearness biframe $(L,\mathcal{U}L)$, $\delta : L \to CFL$ is the universal Cauchy bifilter on $(L,\mathcal{U}L)$.

PROOF. A proof similar to that of Proposition 5.5 applies: replace the nucleus n by m (see Lemma 3.3) and omit all reference to regularity and the condition (N3). ∎

6 Completeness and convergence

Quasi-completeness for quasi-nearness biframes is defined in terms of certain strict surjections being isomorphisms (see Definition 2.3). This is the point-free analogue of the well-known fact that a complete uniform space is closed in every uniform space in which it is embedded. In this section we establish the appealing result that quasi-completeness can in fact be regarded in terms of the convergence of certain Cauchy bifilters, namely the regular ones. Moreover, to test for quasi-completeness, one need only check whether or not the universal regular Cauchy bifilter σ converges.

Definition 6.1 A bifilter $\varphi : L \to T$ *converges* if there is a biframe map $h : L \to T$ with $h \leq \varphi$.

A justification for the definition of convergence is as follows: In a bispace, a bifilter $\mathcal{F}$ converges to a point x if $\mathcal{N}_x \subseteq \mathcal{F}$, where $\mathcal{N}_x$ is the set of neighbourhoods of x in the join topology. (This latter is indeed a bifilter.) In the point-free setting, a bifilter F, consisting of elements of the given biframe L, converges if there is a completely prime filter, P, with $P \subseteq F$. (In this setting, a bifilter P on L is completely prime if P, viewed as a filter on L_0, is completely prime.) Such a completely prime filter P gives rise via its characteristic function to a biframe map $\varphi_P : L \to \mathbf{2}$. One then sees that $\varphi_P \leq \varphi_F$ (where φ_F is the characteristic function of F) which motivates our definition for convergence of general bifilters. This discussion is presented with more details in [13].

Theorem 6.2 Let $(L, \mathcal{U}L)$ be a quasi-nearness biframe. The following conditions are equivalent:

1. $(L, \mathcal{U}L)$ is quasi-complete.
2. Every regular Cauchy bifilter on $(L, \mathcal{U}L)$ converges.
3. The universal regular Cauchy bifilter $\sigma : L \to CL$ converges.

PROOF. (1) $\Rightarrow$ (2): Let $\varphi : (L, \mathcal{U}L) \to T$ be a regular Cauchy bifilter on $(L, \mathcal{U}L)$. By the universality of σ, there is a unique biframe map $\widetilde{\varphi} : CL \to T$ with $\widetilde{\varphi}\sigma = \varphi$. Since $(L, \mathcal{U}L)$ is quasi-complete, its quasi-completion γ is an isomorphism, and hence so is σ. Then $\widetilde{\varphi}\sigma$ is a biframe map and $\widetilde{\varphi}\sigma = \varphi$, so certainly $\widetilde{\varphi}\sigma \leq \varphi$, as required.
(2) $\Rightarrow$ (3): Clear.
(3) $\Rightarrow$ (1): Suppose that there exists a biframe map $h : L \to CL$ with $h \leq \sigma$. Then $f = \gamma h$ is a biframe map. For $a \in L_0$, $f(a) = \gamma(h(a)) \leq \gamma\sigma(a) = a$. For $a \in L_i$, take $x \lhd_i a$. Then $x \prec_i a$, so there exists $t \in L_k\, (k \neq i)$ with $x \wedge t = 0, t \vee a = 1$. Now $x \wedge f(t) \leq x \wedge t$ by the calculation above, so $x \wedge f(t) = 0$. However, $f(t) \vee f(a) = 1$, so $x \prec_i f(a)$, and so $x \leq f(a)$. But $a = \bigvee\{x \in L_i : x \lhd_i a\}$, so we obtain $a \leq f(a)$. This means that f agrees with the identity map on $L_1 \cup L_2$, so $\gamma h = f = \mathrm{id}$.
Then $\gamma h \gamma = \gamma$, and since γ is monic (all biframes in question being regular) this gives $h\gamma = \mathrm{id}$. So γ is a biframe isomorphism. This means, of course, that σ is its inverse. Now since σ generates the quasi-nearness on CL it is also a uniform map, hence a uniform isomorphism. Thus $(L, \mathcal{U}L)$ is quasi-complete. ∎

Theorem 6.2 has an analogue for Cauchy bifilters, which we present below.

Theorem 6.3 Let $(L, \mathcal{U}L)$ be a quasi-nearness biframe. The following conditions are equivalent:

1. The map $g : CFL \to L$ is a biframe isomorphism.
2. Every Cauchy bifilter on $(L, \mathcal{U}L)$ converges.
3. The universal Cauchy bifilter $\delta : L \to CFL$ converges.

PROOF. The proof of the previous proposition can be used, with Cauchy bifilters replacing regular Cauchy bifilters, g replacing γ, δ replacing σ and CFL replacing CL. ∎

7 Questions of functoriality

In this section we compare the Cauchy filter quotient, CFL, and the quasi-completion, CL, for a quasi-nearness biframe $(L, \mathcal{U}L)$. Consider the following biframe diagram:

$$\begin{tikzcd}
\mathbb{D}L \arrow[r, "m"] \arrow[dr, "n"'] \arrow[rr, bend left, "\bigvee"] & CFL \arrow[r, shift right, "g"'] \arrow[d, "\tau"] & L \arrow[l, dashed, shift right, "\delta"'] \arrow[dl, dashed, shift left, "\sigma"] \\
 & CL \arrow[ur, shift left, "\gamma"] &
\end{tikzcd}$$

Here τ is the biframe quotient that corresponds to imposing the condition (N3): to be specific, τ is the unique biframe map such that $\tau m = n$, given by Lemma 2.6.
We note further that:

- $\gamma\tau = g$ because $(\gamma\tau)m = \gamma n = \bigvee = gm$ and m is onto.
- If λ is the right adjoint of τ, $\tau\delta = \sigma$ because $\lambda\sigma = \delta$, so $\sigma = \tau(\lambda\sigma) = \tau\delta$.

Proposition 7.1 The Cauchy filter quotient construction, CF, provides a functor from **QNearBiFrm** to **BiFrm**.

PROOF. We have already considered CFL as a quotient of $(\mathbb{D}L)_0$ as in Definition 5.2. This will provide the action of CF on objects. It remains to consider the action of CF on arrows:
Let $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ be a uniform map between quasi-nearness biframes. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{D}M \arrow[r, "m_M"] \arrow[d, "\mathbb{D}h"'] & CFM \arrow[r, "g_M"] \arrow[d, dashed] & M \arrow[d, "h"] \\
\mathbb{D}L \arrow[r, "m_L"'] & CFL \arrow[r, "g_L"'] & L
\end{tikzcd}$$

The outer rectangle commutes because $gm = \bigvee$ and $\bigvee$ is a natural transformation from $\mathbb{D}$ to the identity functor.
We apply Lemma 2.6 to get a biframe map from CFM to CFL as follows: $\ker(m_M)$ has generating pairs $\{(\bigcup \widehat{C}^s, \downarrow 1) : C \in \mathcal{U}M\}$. For any such C we have:

$$\begin{aligned}
m_L\mathbb{D}h(\bigcup \widehat{C}^s) &= m_L\mathbb{D}h(\bigcup\{\downarrow c \cap \downarrow \widetilde{c} : (c,\widetilde{c}) \in C\}) \\
&= m_L(\bigcup\{\downarrow h(c) \cap \downarrow h(\widetilde{c}) : (c,\widetilde{c}) \in C\} \\
&= m_L(\bigcup \widehat{h[C]}^s) \\
&= \downarrow 1 \\
&= m_L\mathbb{D}h(\downarrow 1)
\end{aligned}$$

The fourth equality uses the fact that h is a uniform map. Now Lemma 2.6 applies, since m_M is onto. This gives a unique biframe map $CFh : CFM \to CFL$ such that $(CFh)m_M = m_L(\mathbb{D}h)$.
Then $g_L(CFh)m_M = g_L m_L(\mathbb{D}h) = hg_M m_M$ and so $g_L(CFh) = hg_M$ since m_M is onto.

In conclusion, the uniform map $h : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ is sent by CF to the biframe map $CFh : CFM \to CFL$ in such a way that the diagram below commutes:

$$\begin{tikzcd} CFM \arrow[r, "g_M"] \arrow[d, "CFh"'] & M \arrow[d, "h"] \\ CFL \arrow[r, "g_L"'] & L \end{tikzcd}$$

It is easy to check that CF preserves identities and composition. ∎

Concluding Remarks:

The fact that CF can be regarded as a functor from the category of quasi-nearness biframes to biframes poses the obvious question whether C, the quasi-completion, has similar functorial properties. The obvious modification of the proof for CF cannot be used, as the map in this new argument would not identify the appropriate generating pairs needed by the factorization lemma. In fact, it is already known in the setting of nearness frames that the completion is not functorial (see Proposition 3.4 in [5]) so the quasi-completion cannot be functorial either.

In [5], the notion of a nearness frame having "enough regular Cauchy filters" (abbreviated by "erc") proved useful in examining the functoriality question. Specifically, in Proposition 2.8 it is shown that, for erc nearness frames, completion is a functor into the category of erc nearness frames and frame homomorphisms. In a subsequent paper we will provide a similar such notion for quasi-nearness biframes and exhibit quasi-completion as a functor.

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